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System of Non-linear equations

For a problem of the form $F(x)=0$, the solⁿ process involves finding the value of ' x ' that will satisfy the equation $F(x)=0$.

• Types of equations

1. Linear Equation

For example, $y = mx + c$

2. Algebraic Equation

For example, $3x + 5y - 21 = 0$ (linear)

$2x + 3xy + 25 = 0$ (non-linear)

3. Transcendental Equation

For example, $2 \sin x - x = 0$

$e^x \sin x + 1 = 0$

$\log x^2 - 1 = 0$

4. Polynomial Equation

For example, $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$

Method of solving non-linear eqⁿ.

i) Direct Analytical method

iv) Iterative method ✓

ii) Graphical method

iii) Trial and error method

Iterative method

It begins with one or more guesses at the solⁿ being sought.

- It generates a sequence of estimate.
- There are two types of methods

- Bracketing → Starts with two initial guesses that bracket the root.
- Non-Bracketing → Use single starting value that don't necessarily bracket the root.

i) Bisection method

- Bracketing method

→ Also known as half interval method.

→ Let $f(x)$ be defined in the interval $[x_n, x_p]$ and $f(x_p)$ are opposite sign such that $f(x_n) * f(x_p) < 0$, then root lies in between x_n & x_p .

• Algorithm.

1. Read x_n & x_p , define stopping criteria E .
2. Calculate $x_m = (x_n + x_p) / 2$.
3. If $\left| \frac{x_p - x_n}{x_n} \right| < E$ go to step ②.
4. Check for $f(x_m)$: if $f(x_m) = 0$ go to step ⑥
 if $f(x_m) < 0$; $x_n = x_m$
 if $f(x_m) > 0$; $x_p = x_m$

5. Repeat steps 2, 3, 4 until ϵ is met.
6. Print x_m .
7. Stop.

Q
Solⁿ Find the root of eqⁿ $x^3 - 2x - 5 = 0$ using Bisection method. ($\epsilon = 0.01$)

$$f(x) = x^3 - 2x - 5 = 0$$

x	-1	0	1	2	3	4	5
$f(x)$	-4	-5	-6	-1	16	51	110

Since, the functional value changes from negative to positive between 2 to 3.
 $x_n = 2, x_p = 3$

n	x_n	x_p	$x_m = \left(\frac{x_n + x_p}{2}\right)$	$f(x_m)$
0	2	3	2.5	5.625 (+ve)
1	2	2.5	2.25	1.89 (+ve)
2	2	2.25	2.125	0.34 (+ve)
3	2	2.125	2.0625	-0.35 (-ve)
4	2.0625	2.125	2.0937	-0.0095 (-ve)
5	2.0937	2.125	2.1094	0.1671 (+ve)
6	2.0937	2.1094	2.1016	0.0790 (+ve)
7	2.0937	2.1016	2.0977	0.0352 (+ve)

$$\text{Error} = \left| \frac{2.1016 - 2.0977}{2.1016} \right| = 0.019$$

$\therefore$ Root is 2.0977

$$\in \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right) \dots \left(\frac{1}{2}\right) \Rightarrow \in \frac{1}{2^n}$$

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Convergence of Bisection method

In this method, we choose a midpoint ' x_m ' in the interval between x_n and x_p . Depending upon the sign of $f(x_m)$, x_p or x_n is set equal to x_m such that root lies in the interval.

In the next iteration, the interval containing the root is reduced by a factor of 2. The same process is repeated for the new interval. If the procedure is repeated ' n ' times, the interval containing the root is reduced by the size $\frac{x_p - x_n}{2^n} = \frac{\Delta x}{2^n}$.

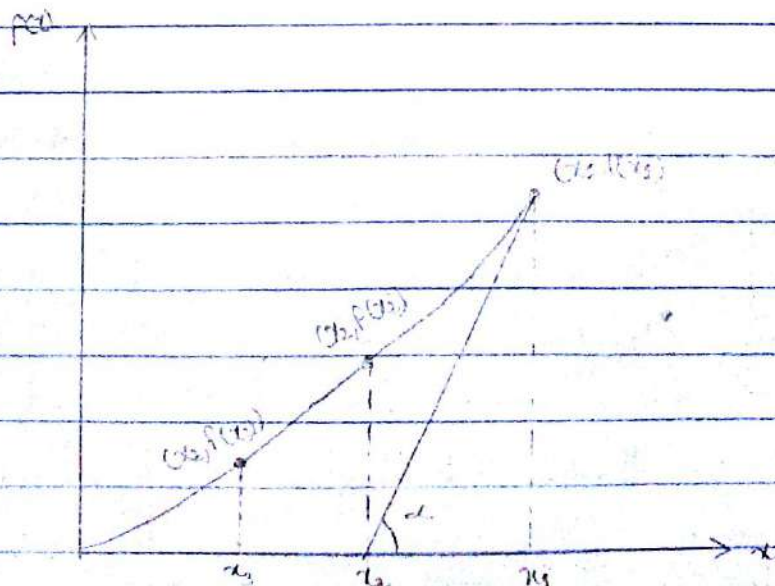
After iteration, the root must lie within $\pm \frac{\Delta x}{2}$ of our estimate. Hence the error bound at n th iteration is

$$E_n = \frac{\Delta x}{2^n}$$

$$\therefore \text{In General, } E_{n+1} = \frac{\Delta x}{2^{n+1}} = \frac{\Delta x}{2^n \cdot 2} = \left(\frac{\Delta x}{2^n}\right) \cdot \frac{1}{2} = \frac{E_n}{2}$$

Since error is reduced to half in each iteration, bisection method is guaranteed to converge.

Newton Raphson Method



From figure,

$$\tan \alpha = \frac{f(x_1)}{x_1 - x_2} = f'(x_1) \quad [\text{where, } f'(x_1) \text{ is the slope of } f(x) \text{ at } x = x_1]$$

$$\text{or, } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

⇒ Algorithm

1. Read an initial value to x' say x_0 and ^{define/} read stopping criteria, ϵ .
2. Evaluate $f(x_0)$ and $f'(x_0)$
if $f(x_0) = 0$ then root will be x_0 .
3. Find improved estimate:
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
4. Check for accuracy: if $\left| \frac{x_1 - x_0}{x_1} \right| < \epsilon$
5. Replace x_0 by x_1 and repeat step (3) and (4)
6. Print x
7. Stop

Q Find the root of $f(x) = x^3 - 3x + 2 = 0$ with initial condition $x = 0$ using Newton Raphson method correct to two decimal places.

Solⁿ Here,

$$x_0 = 0$$

$$f(x) = x^3 - 3x + 2$$

$$f'(x) = 3x^2 - 3$$

$$f(x_0) = 2$$

$$f'(x_0) = -3$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.6667$$

n	x_0	$f(x_0)$	$f'(x_0)$	x_1
0	0	2	-3	0.667
1	0.667	0.296	-1.6665	0.8444
2	0.8444	0.0688	-0.8609	0.9243
3	0.9243	0.0168	-0.4370	0.9627
4	0.9627	0.0041	-0.2196	0.9811
5	0.9811	0.0011	-0.1123	0.9909
6	0.9909	0.0002	-0.0544	0.9946
7	0.9946	0.0001	-0.0323	0.9977

Since $x_n = x_{n+1}$ upto ^{two} three decimal place.

Hence, the solⁿ is 0.99

Convergence of Newton Raphson Method

let x_n be the estimate root of a fcn f(x). If x_n and x_{n+1} are close to each other then using Taylor's Series expansion,

$$f(x_{n+1}) = f(x_n) + f'(x_n) (x_{n+1} - x_n) + \frac{f''(x_n)}{2} (x_{n+1} - x_n)^2 + \dots \quad \text{--- (i)}$$

let us assume that the exact root of f(x) be x_r , then

$$x_{n+1} = x_r \Rightarrow f(x_{n+1}) = f(x_r) = 0$$

$$\therefore 0 = f(x_n) + f'(x_n) (x_r - x_n) + \frac{f''(x_n)}{2} (x_r - x_n)^2 + \dots \quad \text{--- (ii)}$$

Now,

we have from N.R method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\text{or, } f(x_n) = (x_n - x_{n+1}) \cdot f'(x_n) \quad \text{--- (iii)}$$

Substituting the value of $f(x_n)$ from eqⁿ (iii) to eqⁿ (ii)

$$0 = (x_n - x_{n+1}) \cdot f'(x_n) + \frac{f''(x_n)}{2} (x_r - x_n)^2 + \dots$$

$$\text{or, } 0 = f'(x_n) [x_n - x_{n+1} + x_r - x_n] + \frac{f''(x_n)}{2} (x_r - x_n)^2 + \dots$$

$$\text{or, } 0 = (x_r - x_{n+1}) \cdot f'(x_n) + \frac{f''(x_n)}{2} (x_r - x_n)^2 + \dots \quad \text{--- (iv)}$$

Now, we know that error in estimate of x_{n+1} is $e_{n+1} = x_r - x_{n+1}$

Similarly the error in estimate of x_n is $e_n = x_s - x_n$

Substituting the values of e_n and e_{n+1} in eqⁿ (10) and neglecting higher terms

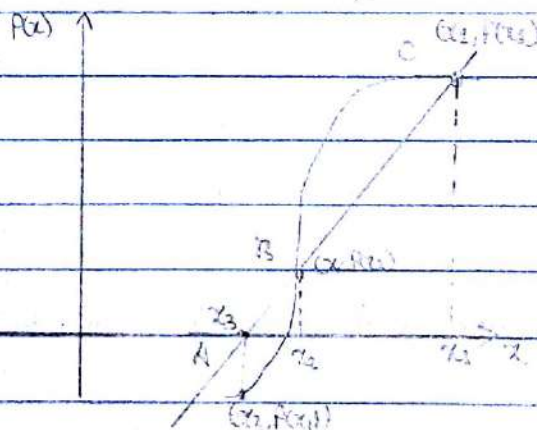
$$0 = e_{n+1} \cdot f'(x_n) + \frac{f''(x_n)}{2} \cdot e_n^2$$

$$\text{or, } e_{n+1} = -\frac{f''(x_n)}{2f'(x_n)} \cdot e_n^2$$

$$\therefore e_{n+1} \propto e_n^2$$

Hence, Newton Raphson method has quadratic convergence.

Secant Method



$$\text{Slope of line AC } m_1 = \frac{f(x_1) - 0}{x_1 - x_3} = \frac{f(x_1)}{x_1 - x_3}$$

$$\text{Slope of line AB } m_2 = \frac{f(x_2) - 0}{x_2 - x_3} = \frac{f(x_2)}{x_2 - x_3}$$

Since,

$$m_1 = m_2$$

(Both slope lies on same plane)

$$\frac{f(x_1)}{x_1 - x_3} = \frac{f(x_2)}{x_2 - x_3}$$

$$\text{or, } (x_2 - x_3) \cdot f(x_1) = f(x_2) \cdot (x_1 - x_3)$$

$$\text{or, } x_2 \cdot f(x_1) - x_3 \cdot f(x_1) = x_1 \cdot f(x_2) - x_3 \cdot f(x_2)$$

$$\text{or, } x_3 (f(x_2) - f(x_1)) = x_1 \cdot f(x_2) - x_2 \cdot f(x_1)$$

$$\therefore x_3 = \frac{x_1 \cdot f(x_2) - x_2 \cdot f(x_1)}{f(x_2) - f(x_1)}$$

• Algorithm.

1. Decide initial points x_1 and x_2 and ϵ .
2. Compute $F_1 = f(x_1)$, $F_2 = f(x_2)$
3. Compute $x_3 = \frac{f_2 x_1 - f_1 x_2}{f_2 - f_1}$
4. Test for accuracy of x_3 :
 if $\left| \frac{x_3 - x_2}{x_3} \right| > \epsilon$ then
 Set $(x_1 = x_2 \text{ and } f_1 = f_2)$
 Set $(x_2 = x_3 \text{ and } f_2 = f_3)$ and go to step 2
 otherwise set root = x_3 .
5. Print root.
6. Stop.

Q

Solve

-1.7417

$$x^2 - 4x - 10 = 0, \quad E = 0.05$$

Solⁿ

let, initial guess be $x_1 = 2, x_2 = 3$

$$x_3 = \frac{x_1 F(x_2) - x_2 F(x_1)}{F(x_2) - F(x_1)} \rightarrow \frac{AB - CA}{B - A}$$

$$= \frac{2 \times (-13) - 3 \times (-14)}{-13 - (-14)}$$

$$= \frac{-26 + 42}{-13 + 14}$$

$$= \frac{16}{1}$$

$$= 16$$

x_1	x_2	x_3	$F(x_1)$	$F(x_2)$	$F(x_3)$
2	3	16	-14	-13	182
3	16	3.8667	-13	182	-10.5154
16	3.8667	4.5294	182	-10.5154	-7.6021
3.8667	4.5294	6.2587	-10.5154	-7.6021	4.1339
4.5294	6.2587	5.6493	-7.6021	4.1339	-0.6826
6.2587	5.6493	5.7356	4.1339	-0.6826	-0.0452
5.6493	5.7356	5.7417	-0.6826	-0.0452	3.1889167
5.7356	5.7417	5.7416	-0.0452	3.1889167	-4.29164

$$E = \left| \frac{x_3 - x_2}{x_3} \right|$$

Since $E < 0.05$

Thus

$$x = 5.74$$

Fixed Point Iterative Method

- Also known as direct substitution method or method of successive approximation.
- Applicable if $f(x)$ can be expressed in the form

$$x = g(x)$$

- If x_0 be the initial guess to the root, then next approx will be

$$x_1 = g(x_0)$$

$$x_2 = g(x_1)$$

$$\vdots$$

$$x_{i+1} = g(x_i)$$

In general,

$$x_{n+1} = g(x_n)$$

Q. Find the square root of 5 using FPI method.

Solⁿ

$$x = \sqrt{5}$$

$$\text{or, } x^2 = 5$$

$$\text{or, } x^2 - 5 = 0 = f(x)$$

$$\Rightarrow x = \frac{5}{x}$$

$$(\because x = g(x))$$

Case I:

$$\text{let } x_0 = 1,$$

$$x_1 = g(x_0) = 5/1 = 5$$

$$x_2 = g(x_1) = 5/5 = 1$$

$$x_3 = g(x_2) = 5$$

$$x_4 = g(x_3) = 1$$

$$\vdots$$

It is known as ~~oscillatory~~ oscillatory divergence

Case II:

Add x to both sides of $f(x)$

$$x = x^2 - 5 + x$$

$$\therefore g(x) = x^2 - 5 + x$$

let, $x_0 = 0$

$$x_1 = g(x_0) = -5$$

$$x_2 = g(x_1) = 15$$

$$x_3 = g(x_2) = 235$$

$$x_4 = g(x_3) = 55455$$

It is known as monotone divergence.

Case III

we have,

$$x = 5/x$$

Add x on both sides,

$$x + x = \frac{5}{x} + x$$

$$\text{or, } 2x = \frac{5+x^2}{x}$$

$$\therefore x = \frac{5+x^2}{2x}$$

$$\therefore g(x) = \frac{5+x^2}{2x}$$

let, $x_0 = 1$

$$x_1 = g(x_0) = 3$$

$$x_2 = g(x_1) = 2.3333$$

$$x_3 = g(x_2) = 2.2381$$

$$x_4 = g(x_3) = 2.2361$$

$$x_5 = g(x_4) = 2.2361$$

It converges at $x = 2.2361$

$$\therefore x = 2.2361$$

Horner's method of finding roots

- Historic name Synthetic Division

- let the true root of $f(x) = 0$ lies in between α and $\alpha + 1$, where α is an
 $\alpha . d_1 d_2 d_3 \dots$

where $d_1, d_2, d_3, \dots$ are digits in their decimal parts.

- To Find the value of d_1 , first diminish the roots of $f(x) = 0$ by α .

For example,

if root lies in between 1 and 2 then root of

$$1 . d_1 d_2 d_3 \dots$$

- Make a transformed eqⁿ by synthetic division

- Multiply the root of eqⁿ by order of 10 & diminish other decimal parts and so on.

- Continue the process to obtain the roots to any desired level of accuracy.

1.570

Q
Solⁿ

$$f(x) = x^3 + 9x - 18$$

→ Correct to two decimal points.

x	-1	0	1	2	3	4
$f(x)$	-28	-18	-8	8	36	82

Since the value of $f(x)$ changes from -ve to +ve
from $x=1$ to $x=2$

$$\therefore \text{Root} = 1.d_1d_2$$

	Coeff. :	x^3	x^2	x^1	x^0
1		1	0	9	-18
		↓	+ 1	+ 1	+ 10
1		1	1	10	(-8)
		↓	+ 1	+ 2	
1		1	2	(12)	
		↓	+ 1		
1		1	(3)		
		↓			
		(1)			

$\therefore$ The transformed eqⁿ is

$$1x^3 + 3x^2 + 12x - 8 = 0$$

Multiplying by order of 10, we get

$$10^0 x^3 + 10^1 \cdot (3x^2) + 10^2 \cdot (12x) - 10^3 (8) = 0$$

$$\text{Or, } x^3 + 30x^2 + 1200x - 8000 = 0$$

x	-1	0	1	2	3	4	5	6	7
$F(x)$	-9171	-8000	-6769	-5472	-4103	-2656	-1125	496	2213

Since, root lies in between 5 and 6

$$\therefore d_1 = 5$$

$\therefore$ Root is $1.5 d_2$

Coeff	x^3	x^2	x^1	x^0
5	1	30	1200	-8000
	↓	+5	+175	+6875
5	1	35	1375	-1125
	↓	+5	200	
5	1	40	1575	
	↓	+5	3	
5	1	45		
	↓			
	①			

$\therefore$ The transformed eqⁿ is

$$1 \cdot x^3 + 45x^2 + 1575x - 1125 = 0$$

Multiplying by orders of 10, we get

$$10^0 x^3 + 10^1 \cdot (45x^2) + 10^2 \cdot (1575x) + 10^3 \cdot 1125 = 0$$

$$\propto x^3 + 450x^2 + 157500x - 1125000 = 0$$

x	-1	0	1	2	3	4	5	6
$f(x)$	-1282051	-1125000	-967049	-808192	-648423	-487736	-326125	-163584

7	8	9
-107	164312	329679

Since root lies in between 7 and 8

$$\therefore d_2 = 7$$

$\therefore$ Root is $1 \cdot d_1 d_2 = 1.57 \text{ } \Phi$

Newton's Method of Polynomial

Consider a polynomial function

$$f_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

Then according to remainder theorem, if we divide $f_n(x)$ by the factor $(x - x_r)$ then we get the Quotient $Q_{n-1}(x)$ of degree $(n-1)$ i.e.

$$Q_{n-1}(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \dots + b_2 x + b_1$$

and Remainder is R .

$$\therefore f_n(x) = (x - x_r) \cdot Q_{n-1}(x) + R$$

Taking initial approximation $x_r = x_0$, we try to improve the value of x_r such that

$$R(x_r) = 0$$

This eq can be solved by using N-R method to get improved value of x_r

$$\therefore x_{k+1} = x_k - \frac{f_n(x_k)}{f'_n(x_k)}$$

• Steps

- Evaluate $f(x_k)$ and $f'(x_k)$ by use of synthetic division
- Use Newton-Raphson method to find the improved value of root.

Q.11
Solve

$$x^3 + x^2 - 3x - 3 = 0 \text{ with } x_0 = 2$$

Coeff	x^3	x^2	x^1	x^0
2	1	1	-3	-3
	↓	+2	+6	+6
2	1	3	3	③ $\rightarrow f(x)$
	↓	+2	+10	
	1	5	⑬ $\rightarrow f'(x)$	

Applying N-R method

$$x_1 = x_0 - \frac{f(x)}{f'(x)}$$

$$= 2 - \frac{3}{13} = 1.7692$$

Coeff	x^3	x^2	x^1	x^0
1.7692	1	1	-3	-3
	↓	+ 1.7692	4.8993	3.3602
1.7692	1	2.7692	1.8993	① 0.3602
	↓	1.7692	8.0293	
	1	4.5384	② 9.9286	

Applying N-R method

$$x_2 = x_1 - \frac{f(x)}{f'(x)}$$

$$= 1.7692 - \frac{0.3602}{9.9286}$$

$$= 1.7329$$

Coeff	x^3	x^2	x^1	x^0
1.7329	1	1	-3	-3
	↓	1.7329	4.7358	3.0080
1.7329	1	2.7329	1.7358	0.0080
	↓	1.7329	8.2067	
	1	4.7358	9.9425	

Applying N-R method

$$x_3 = x_2 - \frac{F(x)}{F'(x)}$$

$$= 1.7329 - \frac{0.0080}{9.9425}$$

$$= 1.7321$$

$\therefore$ Root is 1.732

②

#

Interpolation

If a function is constructed such that it passes through all the data points, this method of constructing a function and estimating the values at non-tabulated points is called interpolation and the function is called interpolating polynomial.

#

Lagrange Polynomial

Let $(x_0, f_0), (x_1, f_1), \dots, (x_n, f_n)$ be the $n+1$ points to derive the formula for polynomial of degree ' n '. Let us consider a second order polynomial of the form

$$P(x) = b_1(x-x_0)(x-x_2) + b_2(x-x_1)(x-x_2) + b_3(x-x_2)(x-x_0) \quad \text{--- (1)}$$

If $(x_0, f_0), (x_1, f_1), (x_2, f_2)$ be three interpolating points:

$$P(x_0) = b_2(x_0-x_1)(x_0-x_2) \Rightarrow f_0 \Rightarrow b_2 = \frac{f_0}{(x_0-x_1)(x_0-x_2)}$$

$$P(x_1) = b_3(x_1-x_2)(x_1-x_0) \Rightarrow f_1 \Rightarrow b_3 = \frac{f_1}{(x_1-x_2)(x_1-x_0)}$$

$$P(x_2) = b_1(x_2-x_0)(x_2-x_1) = f_2 \Rightarrow b_1 = \frac{f_2}{(x_2-x_0)(x_2-x_1)}$$

Substituting the values of b_1, b_2 and b_3 in eqⁿ (1)

$$P(x) = \frac{f_2}{(x_2-x_0)(x_2-x_1)} (x-x_0)(x-x_1) + \frac{f_0}{(x_0-x_1)(x_0-x_2)} (x-x_1)(x-x_2) + \frac{f_1}{(x_1-x_2)(x_1-x_0)} (x-x_2)(x-x_0)$$

$$= \frac{F_0 (x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{F_1 (x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)} + \frac{F_2 (x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$P(x) = F_0 l_0 + F_1 l_1 + F_2 l_2$$

where,

$$l_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$l_1 = \frac{(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)}$$

$$l_2 = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

i.e. In general, for three point

$$\int \frac{2}{x^2} x^2 = (1)^2 x(2)^2 = 4$$

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^2 \frac{(x-x_j)}{(x_i-x_j)}$$

For $n+1$ points,

$$P(x) = \sum_{i=0}^n F_i l_i(x)$$

where,

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{(x-x_j)}{(x_i-x_j)}$$

10

Find the lagrange interpolation polynomial and find the value of $P(x)$ at $x=1.5$
Also find $e^{1.5}$

i	0	1	2	3
x_i	0	1	2	3
$P(x) = e^{x_i} - 1$	0	1.7183	6.3891	19.0855

$$\text{let, } P(x) = P(x)$$

$$l_0 = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{(x-1)(x-2)}{2}$$

$$l_1 = \frac{(x-0)(x-2)}{(1-2)(1-0)} = -x(x-2)$$

$$l_2 = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{x(x-1)}{2}$$

$$P(x) = f_0 l_0 + f_1 l_1 + f_2 l_2$$

$$= 0 \times \frac{(x-1)(x-2)}{2} - 1.7183 \times x(x-2) + 6.3891 \times \frac{x(x-1)}{2}$$

$$\therefore P(1.5) = -1.7183 \times 1.5 \times (1.5-2) + \frac{6.3891 \times 1.5 \times (1.5-1)}{2}$$

$$= 3.6846$$

we have,

$$P(1.5) = e^{1.5} - 1 = 3.6846$$

$$\Rightarrow e^{1.5} = 3.6846 + 1 = 4.6846$$

Finite Difference Method

- Interpolation for equally spaced data.

- If we have a table of values (x_i, y_i) of any function $y=f(x)$ such that the value of x is being equally spaced i.e. $x_i = x_0 + ih$, the method of finding the solution of $f(x)$ for some ' x ' in the range $x_0 \leq x \leq x_n$ is based on concept of Finite difference.

• Two methods:

i) Forward difference method

ii) Backward " "

i) Forward difference method:

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	f_0	$\Delta f_0 = f_1 - f_0$	$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$	$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$
$x_0 + h$	f_1	$\Delta f_1 = f_2 - f_1$	$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$	$\vdots$
$x_0 + 2h$	f_2	$\Delta f_2 = f_3 - f_2$	$\vdots$	
$x_0 + 3h$	f_3	$\vdots$	$\vdots$	
$\vdots$	$\vdots$	$\vdots$		
$x_0 + nh$	f_n			

Fig: Difference table

Now, the polynomial is defined by Newton-Gregory formula:

$$p(s) = f_0 + \frac{s \Delta f_0}{1!} + \frac{s(s-1) \Delta^2 f_0}{2!} + \frac{s(s-1)(s-2) \Delta^3 f_0}{3!} + \dots$$

where, $s = \frac{x - x_0}{h}$

x	0.1	0.2	0.3	0.4
$f(x)$	1.005	1.020	1.045	1.081

Find $f(x)$ at $x = 0.16$.

Solⁿ

Constructing the difference table,

x	$f(x) \rightarrow f_0$	$\Delta f \rightarrow \Delta f_0$	$\Delta^2 f$	$\Delta^3 f$
0.1	1.005	0.0150	0.0100	0.0010
0.2	1.020	0.0250	0.0110	
0.3	1.045	0.0360		
0.4	1.081			

$$h = x_1 - x_0 = 0.1$$

$$\therefore s = \frac{x - x_0}{h} = \frac{0.16 - 0.1}{0.1} = 0.6$$

Newton-Gregory Formula:

$$P(0.16) = f_0 + \frac{s \Delta f_0}{1!} + \frac{s(s-1) \Delta^2 f_0}{2!} + \frac{s(s-1)(s-2) \Delta^3 f_0}{3!}$$

$$= 1.005 + \frac{0.6 \times 0.0150}{1!} + \frac{0.6(0.6-1) \times 0.0100}{2!} + \frac{0.6(0.6-1)(0.6-2)}{3!} \times 0.0010$$

$$= 1.005 + 0.0090 - 0.0012 + 0.0001$$

$$= 1.0029 = f(x)$$

when the value lies in front of table $\rightarrow$ forward

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Q.11)

Backward difference method

- Construct the difference table exactly similar to that of forward difference method

- Newton - Gregory formula is given by

$$P_n(s) = f_n + s \Delta f_n + s(s+1) \Delta^2 f_n + s(s+1)(s+2) \Delta^3 f_n + \dots$$

$\frac{1! \qquad 2! \qquad 3!}{}$

where, $s = \frac{x - x_n}{h}$

Q.

x	10	20	30	40	50
$f(x)$	0.1736	0.342	0.5	0.6428	0.768

Find $f(45) = ?$

Sol.

Constructing the difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
10	0.1736	0.1684	-0.0104	-0.1416	0.1384
20	0.342	0.1580	-0.0152	-0.0024	
30	0.5	0.1428	-0.0176		
40	0.6428	0.1252			
50	0.768				

f_n

$$h = x_1 - x_0 = 20 - 10 = 10$$

$$s = \frac{x - x_n}{h} = \frac{45 - 50}{10} = -0.5$$

Newton - Gregory formula,

$$P_n(45) = f_n + \frac{s \Delta f_n}{1!} + \frac{s(s+1)}{2!} \Delta^2 f_n + \frac{s(s+1)(s+2)}{3!} \Delta^3 f_n + \frac{s(s+1)(s+2)(s+3)}{4!} \Delta^4 f_n$$

$$= 0.768 + \frac{(-0.5) \times 0.1252}{1!} + \frac{(-0.5)(-0.5+1)}{2!} \times (-0.0176) +$$

$$\frac{(-0.5)(-0.5+1)(-0.5+2)}{3!} \times (-0.0024) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!} \times 0.1392$$

$$= 0.768 - 0.0626 + 0.00110 + 0.00015 - 0.00544$$

$$= 0.70121$$

$$\therefore f^{45} = 0.70121$$

Divided Difference Method

- Tabulated values need not be equally spaced.
- Newton's Form of polynomial is used:

$$F(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1}) \quad \text{--- (1)}$$

where,

coefficients $a_0, a_1, a_2, \dots, a_n$ can be calculated using divided difference table

	x_i	f_i	IDD	IID	IID
	x_0	f_0	$F(x_0, x_1) = \frac{f_1 - f_0}{x_1 - x_0}$	$F(x_0, x_1, x_2) = \frac{F(x_1, x_2) - F(x_0, x_1)}{x_2 - x_0}$	
	x_1	f_1	$F(x_1, x_2) = \frac{f_2 - f_1}{x_2 - x_1}$	$F(x_1, x_2, x_3) = \frac{F(x_2, x_3) - F(x_1, x_2)}{x_3 - x_1}$	
	x_2	f_2	$F(x_2, x_3) = \frac{f_3 - f_2}{x_3 - x_2}$		
	x_3	f_3	$F(x_3, x_4) = \frac{f_4 - f_3}{x_4 - x_3}$		
	x_4	f_4			
III D					
$F(x_0, x_1, x_2, x_3) = \frac{F(x_1, x_2, x_3) - F(x_0, x_1, x_2)}{x_3 - x_0}$					

Q

Find $P(x)$ at $x = 2.5$

x	-3	-1	0	3	5
$F(x)$	-30	-22	-12	330	3458

Sol

Constructing the divided difference table

x_i	f_i	I DD	II DD	III DD	IV DD
-3	-30 $\xrightarrow{a_0}$	$\frac{-22+30}{-1+3} = 4 \xrightarrow{a_1}$	2	4	5
-1	-22	10	26	44	
0	-12	114	290		
3	330	1564			
5	3458				

Newton's form of polynomial,

$$F(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$\begin{aligned} \therefore F(2.5) &= -30 + 4(2.5+3) + 2(2.5+3)(2.5+1) + 4(2.5+3)(2.5+1)(2.5+0) \\ &\quad + 5(2.5+3)(2.5+1)(2.5-0)(2.5-2) \\ &= 102.6875 \end{aligned}$$

Regression Analysis / Curve Fitting

The process of establishing a relationship in the particular form of mathematical equation from the experimental data is known as regression analysis or curve fitting.

We fit the data under following conditions:

- i) Relation is linear
- ii) " " transcendental
- iii) " " polynomial.

Q Fitting linear equation

$$\text{let, } y = f(x) = a + bx$$

Then minimizing the errors while fitting the straight line from experimental data, we get the regression equation as:

$$\left. \begin{aligned} \sum y_i &= na + b \sum x_i \\ \sum x_i y_i &= a \sum x_i + b \sum x_i^2 \end{aligned} \right\} \text{--- ①}$$

It can be expressed in matrix form as:

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

Q Fit a st. line to the following data

x	1	2	3	4	5
y	3	4	5	6	8

Given, no. of datas (n) = 5

The straight line is $y = a + bx$

The linear regression expressed in matrix form is

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix} \quad \text{--- (i)}$$

x_i	y_i	x_i^2	$x_i y_i$
1	3	1	3
2	4	4	8
3	5	9	15
4	6	16	24
5	8	25	40
$\sum x_i = 15$	$\sum y_i = 26$	$\sum x_i^2 = 55$	$\sum x_i y_i = 90$

From (i)

$$\begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 26 \\ 90 \end{bmatrix}$$

$$\text{i.e. } 5a + 15b = 26 \quad \text{--- (i)}$$

$$15a + 55b = 90 \quad \text{--- (ii)}$$

Solving (i) & (ii),

$$\therefore a = 1.6$$

$$\therefore b = 1.2$$

$\therefore$ Required st. line is $y = 1.6 + 1.2x$

ii) Fitting polynomial data

The polynomial of degree 'm' is given as

$$y = f(x) = a_1 + a_2 x + a_3 x^2 + \dots + a_m x^{m-1}$$

and second order polynomial is given as:

$$y = a_1 + a_2 x + a_3 x^2$$

and its regression can be expressed in matrix form as:

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}$$

Q. Fit a second order polynomial.

x	1	2	3	4
y	6	11	18	27

Solⁿ Given no. of data, $n=4$

The second order polynomial is given as

$$y = a_1 + a_2 x + a_3 x^2$$

The regression expressed in matrix form is

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix} \quad \text{--- (1)}$$

x_i	y_i	x_i^2	x_i^3	x_i^4	$x_i y_i$	$x_i^2 y_i$
1	6	1	1	1	6	6
2	11	4	8	16	22	44
3	18	9	27	81	54	162
4	27	16	64	256	108	432
$\Sigma x_i = 10$	$\Sigma y_i = 62$	$\Sigma x_i^2 = 30$	$\Sigma x_i^3 = 100$	$\Sigma x_i^4 = 354$	$\Sigma x_i y_i = 190$	$\Sigma x_i^2 y_i = 644$

From (i)

$$\begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 62 \\ 190 \\ 644 \end{bmatrix}$$

$$\text{i.e. } 4a_1 + 10a_2 + 30a_3 = 62 \quad \text{--- (i)}$$

$$10a_1 + 30a_2 + 100a_3 = 190 \quad \text{--- (ii)}$$

$$30a_1 + 100a_2 + 354a_3 = 644 \quad \text{--- (iii)}$$

Solving (i), (ii) & (iii)

$$\therefore a_1 = 3, \quad a_2 = 2, \quad a_3 = 1$$

Required second order polynomial is

$$y = 3 + 2x + x^2$$

Fitting transcendental equation

The transcendental eqⁿ can be respected as

$$P = P_0 e^{kt}$$

Generally, we write

$$y = ax^b \quad \text{--- (1)}$$

Taking natural log on both sides

$$\begin{aligned} \ln y &= \ln(ax^b) \\ &= \ln a + b \ln x \end{aligned}$$

Comparing this eqⁿ with the eqⁿ of ^{st. line} linear regression i.e. $y = a + bx$

$$y \rightarrow \ln y$$

$$a \rightarrow \ln a$$

$$b \rightarrow b$$

$$x \rightarrow \ln x$$

∴ The linear regression eqⁿ transcendental eqⁿ is

$$\begin{bmatrix} n & \sum \ln x_i \\ \sum \ln x_i & \sum (\ln x_i)^2 \end{bmatrix} \begin{bmatrix} \ln a \\ b \end{bmatrix} = \begin{bmatrix} \sum \ln y_i \\ \sum (\ln x_i)(\ln y_i) \end{bmatrix}$$

Q

Fit a power function of the form $y = ax^b$

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

Solⁿ

Given, no. of data = 5

The regression expressed in matrix form is

$$\begin{bmatrix} n & \sum \ln x_i \\ \sum \ln x_i & \sum (\ln x_i)^2 \end{bmatrix} \begin{bmatrix} \ln a \\ b \end{bmatrix} = \begin{bmatrix} \sum \ln y_i \\ \sum \ln x_i \ln y_i \end{bmatrix} \quad \text{--- (a)}$$

x	y	$\ln x_i$	$(\ln x_i)^2$	$\ln y_i$	$\ln x_i \ln y_i$
1	0.5	0	0	-0.6931	0
2	2	0.6931	0.4805	0.6931	0.4804
3	4.5	1.0986	1.2069	1.5041	1.6524
4	8	1.3863	1.9218	2.0784	2.8827
5	12.5	1.6094	2.5903	2.5257	4.0649
		$\sum \ln x_i =$ 4.7874	$\sum (\ln x_i)^2 =$ 6.1995	$\sum \ln y_i =$ 6.1092	$\sum \ln x_i \ln y_i =$ 9.0804

From (a),

$$\begin{bmatrix} 5 & 4.7874 \\ 4.7874 & 6.1995 \end{bmatrix} \begin{bmatrix} \ln a \\ b \end{bmatrix} = \begin{bmatrix} 6.1092 \\ 9.0804 \end{bmatrix}$$

$$\text{i.e. } 5 \ln a + 4.7874b = 6.1092 \quad \text{--- (i)}$$

$$4.7874 \ln a + 6.1995b = 9.0804 \quad \text{--- (ii)}$$

Solving ① & ②, we get

$$\therefore \ln a = -0.6929 \quad \text{--- ③}$$

$$\therefore b = 1.9998$$

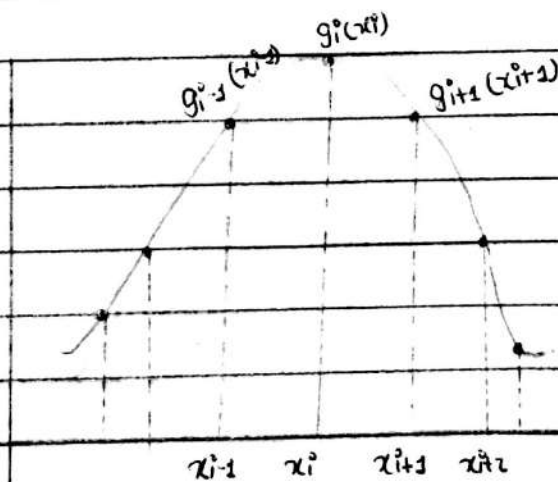
From ③

$$a = \ln^{-1}(-0.6929)$$

$$= 0.5001$$

$\therefore$ The required power function is $y = ax^b$
 $= 0.5001 \cdot x^{1.9998}$

Spline interpolation



Let us consider $(n+1)$ sample points with n number of different sections as shown in figure.

Suppose $g_i(x)$ represents a cubic polynomial between $[x_i, f(x_i)]$ and $[x_{i+1}, f(x_{i+1})]$

then it is given as

$$g_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i \quad \text{--- ①}$$

For cubic polynomial, we have the coefficients

$$a_i = \frac{S_{i+1} - S_i}{6h_i}, \quad b_i = \frac{S_i}{2}, \quad c_i = \frac{f(x_{i+1}) - f(x_i) - a_i h_i^2 - b_i h_i}{h_i}, \quad d_i = f(x_i)$$

Eqⁿ ① can be transformed into

$$h_i S_{i-1} + 2(h_{i-1} + h_i) S_i + h_i S_{i+1} = 6 \left(\frac{f_{i+1} - f_i}{h_i} - \frac{f_i - f_{i-1}}{h_{i-1}} \right)$$

where,

S_i is the spline interpolation constant.
→ needs to be computed

For natural cubic splines,

$$S_0 = S_n = 0$$

Find $f(0.66)$ and $f(2)$ from the following data by spline interpolation.

x	x_0 0	x_1 1	x_2 1.5	x_3 2.25
$f(x)$	2	4.4366	6.7134	13.9130

Since, these are 4-points, ^{i.e. $n=4$} hence there are three cubic splines.

From table

$$h_0 = x_1 - x_0 = 1 - 0 = 1$$

$$h_1 = x_2 - x_1 = 1.5 - 1 = 0.5$$

$$h_2 = x_3 - x_2 = 2.25 - 1.5 = 0.75$$

For natural cubic splines

$$S_0 = S_1 = 0$$

$$S_0 = S_3 = 0$$

From eqⁿ of cubic splines given as:

$$h_i S_{i-1} + 2(h_{i-1} + h_i) S_i + h_i S_{i+1} = 6 \left(\frac{f_{i+1} - f_i}{h_i} - \frac{f_i - f_{i-1}}{h_{i-1}} \right)$$

For $i=1$,

$$h_1 S_0 + 2(h_0 + h_1) S_1 + h_1 S_2 = 6 \left(\frac{f_2 - f_1}{h_1} - \frac{f_1 - f_0}{h_0} \right)$$

$$\text{or, } 0.5 \times 0 + 2(1 + 0.5) S_1 + 0.5 S_2 = 6 \left(\frac{6.7134 - 4.4366}{0.5} - \frac{4.4366 - 2}{1} \right)$$

$$\text{or, } 3 S_1 + 0.5 S_2 = 12.7020 \quad \text{--- (a)}$$

For $i=2$,

$$h_2 S_1 + 2(h_1 + h_2) S_2 + h_2 S_3 = 6 \left(\frac{f_3 - f_2}{h_2} - \frac{f_2 - f_1}{h_1} \right)$$

$$\text{or, } 0.75 S_1 + 2(0.5 + 0.75) S_2 + 0.75 \times 0 = 6 \left(\frac{12.9130 - 6.7134}{0.75} - \frac{6.7134 - 4.4366}{0.5} \right)$$

$$\text{or, } 0.75 S_1 + 2.5 S_2 = 30.2752 \quad \text{--- (b)}$$

Solving (a) and (b), we get

$$S_1 = 2.3323$$

$$S_2 = 11.4104$$

Now,

We have,

$$a_i = \frac{S_{i+1} - S_i}{6h_i}$$

$$\therefore a_0 = \frac{S_1 - S_0}{6h_0} = \frac{2.3323 - 0}{6 \times 1} = 0.3887$$

$$\therefore a_2 = \frac{S_3 - S_2}{6h_2} = \frac{0 - 11.4104}{6 \times 0.75} = -2.5356$$

$$b_i = S/2$$

$$\therefore b_0 = S_0/2 = 0/2 = 0$$

$$\therefore b_2 = S_2/2 = 11.4104/2 = 5.7052$$

$$c_i = \frac{f(x_{i+1}) - f(x_i) - a_i h_i^2 - b_i h_i}{h_i}$$

$$\begin{aligned} c_0 &= \frac{f(x_{0+1}) - f(x_0) - a_0 h_0^2 - b_0 h_0}{h_0} \\ &= \frac{f(x_1) - f(x_0) - 0.3887 \times (1)^2 - 0 \times 1}{1} \end{aligned}$$

$$= 4.4366 - 2 - 0.3887 = 2.0479$$

$$c_2 = \frac{f(x_{2+1}) - f(x_2) - a_2 h_2^2 - b_2 h_2}{h_2}$$

$$= \frac{f(x_3) - f(x_2) + 2.5356 \times (0.75)^2 - 5.7052 \times 0.75}{0.75}$$

$$= \frac{f(2.25) - f(1.5) + 1.4263 - 4.2789}{0.75}$$

$$= \frac{13.9130 - 6.7134 + 1.4263 - 4.2789}{0.75} = 5.7966$$

$$d_i^0 = f(x_i^0)$$

$$\therefore d_0 = f(x_0) = f(0) = 2$$

$$\therefore d_2 = f(x_2) = f(1.5) = 6.7134$$

We know,

$$g_i(x) = a_i (x-x_i)^3 + b_i (x-x_i)^2 + c_i (x-x_i) + d_i$$

$$\text{For } i=0,$$

$$\begin{aligned} \therefore g_0(x) &= a_0 (x-x_0)^3 + b_0 (x-x_0)^2 + c_0 (x-x_0) + d_0 \\ &= 0.3887 \times (x-0)^3 + 0 \times (x-0)^2 + 2.0479 (x-0) + 2 \\ &= 0.3887x^3 + 2.0479x + 2 \end{aligned}$$

$$\therefore g_0(0.66) = 0.3887 \times (0.66)^3 + 2.0479 \times 0.66 + 2 = 3.4634$$

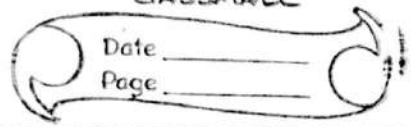
$$\text{For } i=2,$$

$$g_2(x) = a_2 (x-x_2)^3 + b_2 (x-x_2)^2 + c_2 (x-x_2) + d_2$$

$$\begin{aligned} \therefore g_2(2) &= -2.5356 (2-1.5)^3 + 5.7052 (2-1.5)^2 + 5.7966 (2-1.5) + 6.7134 \\ &= 10.7178 \quad \text{Q} \end{aligned}$$

(3)

classmate



#

Numerical Integration and Differentiation

#

Numerical Integration

It is defined as

$$I = \int_a^b y \cdot dx$$

and gives the area of curve under y from $x=a$ to $x=b$.

let, $f(x) = P_n(x)$ be the n th order polynomial that interpolates $(n+1)$ points.

Then the result of integrand I is called $(n+1)$ Newton-Cotes formula.

let us consider a general Newton interpolation formula

$$f(x) = f_0 + s \Delta f_0 + \frac{s(s-1)}{2!} \Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!} \Delta^3 f_0 + \dots$$

where, $x = a + sh$

$$\therefore \frac{dx}{ds} = h$$

$$\text{or, } dx = h \cdot ds \quad \text{--- (1)}$$

When $x = a \Rightarrow s = 0$

When $x = b = a + nh \Rightarrow s = n$

Then,

$$I = \int_a^b f(x) \cdot dx$$

$$= \int_0^n \left\{ f_0 + s \Delta f_0 + \frac{s(s-1)}{2!} \Delta^2 f_0 + \dots \right\} \cdot h \cdot ds$$

$$= h \cdot \left[s f_0 + \frac{s^2}{2} \Delta f_0 + \frac{(s^3 - s^2)}{3!} \Delta^2 f_0 + \frac{(s^4 - 3s^3 + 3s^2)}{4!} \Delta^3 f_0 + \dots \right] \Big|_0^n$$

$$= h \left[n f_0 + \frac{n^2}{2} \Delta f_0 + \frac{(n^3 - n^2)}{3!} \Delta^2 f_0 + \frac{(n^4 - n^3 + n^2)}{4!} \Delta^3 f_0 + \dots \right]$$

which is known as General Newton Cotes formula and

$$h = \frac{x_n - x_0}{n}$$

① Trapezoidal Rule (2-point formula)

When $n=1$, in General Newton Cotes formula and neglecting 2nd and higher order differences gives two point formula known as Trapezoidal rule.

$$I = h \left[f_0 + \frac{1}{2} \Delta f_0 \right]$$

$$= h \left[f_0 + \frac{1}{2} (f_1 - f_0) \right]$$

$$= h \left[\frac{2f_0 + f_1 - f_0}{2} \right]$$

$$I = h \left[\frac{f_0 + f_1}{2} \right]$$

Since, this is an expression for area of trapezoid, hence called Trapezoidal Rule.

Q Evaluate Integral $\int_a^b x^3 + 1 \, dx$ for i) $[1, 2]$ ii) $[1, 1.5]$ by Trapezoidal Rule.

Solⁿ

$$I = \int_a^b (x^3 + 1) \cdot dx$$

$$f(x) = x^3 + 1$$

$$x_0 = 1, x_n = 2$$

i) $h = 2 - 1 = 1$

$$f_0 = f(x_0) = 1^3 + 1 = 2$$

$$f_n = f(x_n) = 2^3 + 1 = 9$$

$$I = h \cdot \left[\frac{f_0 + f_n}{2} \right] = 1 \cdot \left[\frac{2 + 9}{2} \right] = 5.5$$

Q.11)

$$x_0 = 1, x_n = 1.5$$

$$h = 1.5 - 1 = 0.5$$

$$f_0 = f(x_0) = 1^3 + 1 = 2$$

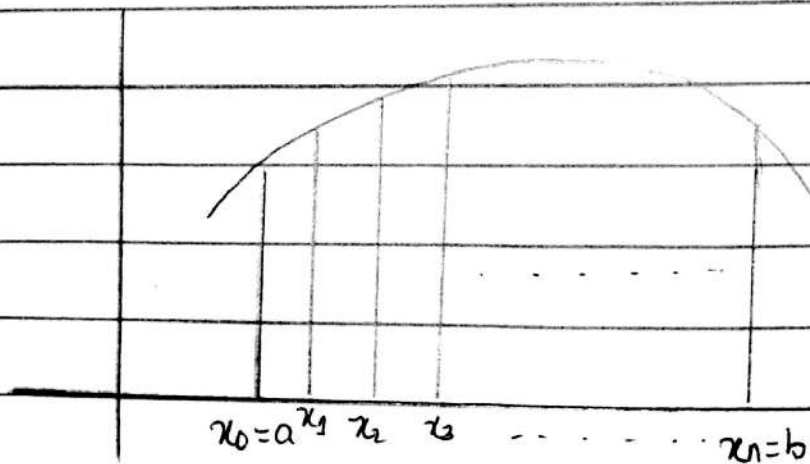
$$f_n = f(x_n) = (1.5)^3 + 1 = 1.125 + 1 = 2.125$$

$$I = h \cdot \left[\frac{f_0 + f_n}{2} \right]$$

$$= 0.5 \cdot \left[\frac{2 + 2.125}{2} \right]$$

$$= 1.5938$$

#

Composite Trapezoidal Rule

$$I = \int_{x_0}^{x_n} f(x) \cdot dx = \int_{x_0}^{x_1} f(x) \cdot dx + \int_{x_1}^{x_2} f(x) \cdot dx + \int_{x_2}^{x_3} f(x) \cdot dx + \dots + \int_{x_{n-1}}^{x_n} f(x) \cdot dx$$

$$= h \left(\frac{f_0 + f_1}{2} \right) + h \left(\frac{f_1 + f_2}{2} \right) + h \left(\frac{f_2 + f_3}{2} \right) + \dots + h \left(\frac{f_{n-1} + f_n}{2} \right)$$

$$= \frac{h}{2} \left[f_0 + f_n + 2f_1 + 2f_2 + 2f_3 + \dots + 2f_{n-1} + 2f_n \right]$$

$$\therefore I = \frac{h}{2} \left[f_0 + f_n + 2(f_1 + f_2 + \dots + f_{n-1}) \right]$$

Q Evaluate $I = \int_0^{\pi} \sin x \cdot dx$ with four strips ($n=4$) using trapezoidal rule.

Solⁿ Here,

$$f(x) = \sin x$$

$$x_0 = 0, x_n = \pi$$

$$h = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

		$0 + \frac{\pi}{4} = \frac{\pi}{4}$	$\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$	$\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
$f(x)$	0	0.707	1	0.707

x	$\frac{3\pi}{4} + \frac{\pi}{4} = \pi$
$f(x)$	0

We know,

$$I = \frac{h}{\lambda} [0 + 0 + 2(0.707 + 1 + 0.707)]$$

$$= \frac{\pi/2}{\lambda} \times \lambda \times 2.4140$$

$$= 3.7919$$

$$\begin{aligned}\Delta(\Delta f_0) &= \Delta(f_1 - f_0) \\ &= (f_2 - f_1) - (f_1 - f_0) \\ &= f_0 - 2f_1 + f_2\end{aligned}$$

Simpson's $\frac{1}{3}$ Rule

Taking $n=2$ in General Newton cotes formula and neglecting third and higher differences we get

$$I = \int_a^b f(x) \cdot dx$$

$$= h \left\{ 2f_0 + \frac{1}{2} \Delta f_0 + \left(\frac{8}{13} - \frac{1}{2} \right) \cdot \frac{\Delta^2 f_0}{2!} \right\}$$

$$= h \left\{ 2f_0 + 2(f_1 - f_0) + \frac{2}{13} \cdot \frac{1}{2} (f_0 - 2f_1 + f_2) \right\}$$

$$= h \left\{ 2f_0 + 2f_1 - 2f_0 + \frac{1}{3} f_0 - \frac{2}{3} f_1 + \frac{1}{3} f_2 \right\}$$

$$= \frac{h}{3} \{ 6f_1 + f_0 - 2f_1 + f_2 \}$$

$$I = \frac{h}{3} \{ f_0 + 4f_1 + f_2 \}$$

Q Evaluate

$$I = \int_{-1}^1 e^x dx \text{ by 3-point Simpson's } \frac{1}{3} \text{ Rule.}$$

Solⁿ Here,

$$h = \frac{b-a}{n} = \frac{1-(-1)}{2} = 1$$

$$f(x) = e^x$$

	x_0	$x_1 = x_0 + h$	$x_2 + h$
x	-1	0	1
$f(x)$	0.3679	1	2.7183

$\uparrow$ $f(x_0) = f_0$ $\uparrow$ f_1 $\uparrow$ f_2

Using Simpson's $\frac{1}{3}$ rule,

$$I = \int_{-1}^1 e^x dx$$

$$= h \{ f_0 + 4f_1 + f_2 \}$$

$$= \frac{1}{3} \{ 0.3679 + 4 \times 1 + 2.7183 \}$$

$$= 2.3621$$

#

Composite Simpson's $\frac{1}{3}$ rule

$$I = \int_{x_0}^{x_n} f(x) dx = \int_{x_0}^{x_1} f(x) dx$$

$$= \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \dots + \int_{x_{n-2}}^{x_n} f(x) dx$$

$$= \frac{h}{3} \{ (f_0 + 4f_1 + f_2) + (f_2 + 4f_3 + f_4) + \dots + (f_{n-2} + 4f_{n-1} + f_n) \}$$

$$\therefore I = \frac{h}{3} \left[f_0 + f_n + 4 \{ f_1 + f_3 + f_5 + \dots \} + 2 \{ f_2 + f_4 + \dots \} \right]$$

Q

Evaluate

$$I = \int_0^{\pi/2} \sqrt{\sin x} \cdot dx \text{ by Simpson's } \frac{1}{3} \text{ rule.}$$

for i) $n=4$ ii) $n=6$ Solⁿ

Here,

$$i) \quad n=4, \quad h = \frac{b-a}{n} = \frac{\pi/2 - 0}{4} = \frac{\pi}{8}$$

$$f(x) = \sqrt{\sin x}$$

	x_0	$x_1 = x_0 + h = 0 + \pi/8$			
x	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$
$f(x)$	0	0.6186	0.8409	0.9612	1
	$\uparrow$ f_0	$\uparrow$ f_1	$\uparrow$ f_2	$\uparrow$ f_3	$\uparrow$ f_4

Using composite Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3} [f_0 + f_4 + 4(f_1 + f_3) + 2f_2]$$

$$= \frac{\pi/8}{3} [0 + 1 + 4(0.6186 + 0.9612) + 2 \times 0.8409]$$

$$= 1.1732$$

ii) $n=6$, $h = \frac{b-a}{n} = \frac{\pi/2 - 0}{6} = \frac{\pi}{12}$

$$f(x) = \sqrt{\sin x}$$

x	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
$f(x)$	0	0.5087	0.7072	0.8409	0.9306	0.9828	1

Using Simpson's $4/3$ rule,

$$I = \frac{h}{3} [f_0 + f_6 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4)]$$

$$= \frac{\pi/12}{3} [0 + 1 + 4(0.5087 + 0.8409 + 0.9828) + 2(0.7072 + 0.9306)]$$

$$= 1.1873$$

Numerical Double Integration

→ Double integration can be evaluated by repeatedly applying the trapezoidal or Simpson's rule.

i) Trapezoidal rule

$$\text{let, } I = \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} f(x, y) \cdot dx \cdot dy$$

$$\text{where, } x_{i+1} = x_i + h, \quad y_{j+1} = y_j + k$$

Applying trapezoidal rule,

$$f(x) = \frac{h}{2} [f_0 + f_1]$$

$$\begin{aligned} I &= \frac{h}{2} \int_{y_j}^{y_{j+1}} [f(x_i, y) + f(x_{i+1}, y)] \cdot dy \\ &= \frac{h}{2} \cdot \frac{k}{2} [f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})] \\ &= \frac{hk}{4} [f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})] \end{aligned}$$

ii) Simpson's $\frac{1}{3}$ rule

$$\text{let, } I = \int_{y_{j-1}}^{y_{j+1}} \int_{x_{i-1}}^{x_{i+1}} f(x, y) \cdot dx \cdot dy$$

$$J = \frac{h}{3} [f_0 + 4f_1 + f_2]$$
$$I = \frac{h}{3} \int_{y_{j-1}}^{y_j} [f(x_{i-1}, y) + 4f(x_i, y) + f(x_{i+1}, y)] \cdot dy$$

$$= \frac{h}{3} \cdot \frac{K}{3} \left[\left\{ F(x_{i-1}^0, y_{j-1}^0) + 4 F(x_{i-1}^0, y_j^0) + F(x_{i-1}^0, y_{j+1}^0) \right\} + \right.$$

1	4	1	y_{j+1}
4	16	4	
1	4	1	y_{j-1}

x_{i-1} x_{i+1}

Similarly, For composite trapezoidal rule

1	2	1
2	4	2
1	2	1

Q

Evaluate

$$I = \int_0^1 \int_0^1 e^{xy} dx dy \text{ using trapezoidal and Simpson's}$$

rule with $h=k=0.5$.

↓

Solⁿ

Here, $f(x,y) = e^{xy}$

$f(x,y)$

$y \backslash x$	0	0.5	1	
0	$e^{0 \cdot 0} = e^0 = 1$	1.6487	2.7182	1 2 1
0.5	$e^{0.5 \cdot 0} = 1.6487$	2.7182	$e^{1 \cdot 0.5} = 4.4816$	2 4 2
1	2.7182	4.4816	7.3890	1 2 1

Using trapezoidal rule,

$$I = \frac{hk}{4} \left[1 + 2.7182 + 7.3890 + 2.7182 + 2 \{ 1.6487 + 4.4816 + 4.4816 + 1.6487 \} + 4 \times 2.7182 \right]$$

$$= \frac{0.5 \times 0.5}{9} \times 49.2194$$

$$= 3.0762 \text{ } \text{g}$$

Using Simpson's $\frac{1}{3}$ rule,

1	4	1
4	16	4
1	4	1

$$I = \frac{hk}{9} \left[1 + 2.7182 + 7.3890 + 2.7182 + 4 \times \{ 1.6487 + 4.4816 + 4.4816 + 1.6487 \} + 16 \times 2.7182 \right]$$

$$= \frac{0.5 \times 0.5}{9} \times 106.3590$$

$$= 2.9544 \text{ } \text{g}$$

Gaussian Integration

Gauss Integration Assumes an approximation of the form

$$I_g = \int_{-1}^1 f(x) \cdot dx = \sum_{i=1}^n w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2) \quad [\text{for } n=2]$$

and the values of w_1, w_2, x_1, x_2 needs to be calculated.

For $f(x) = 1$ [$f(x_1) = f(x_2) = 1$] :

$$w_1 + w_2 = \int_{-1}^1 f(x) \cdot dx = \int_{-1}^1 1 \cdot dx = x \Big|_{-1}^1$$

$$= 1 - (-1) = 2 \quad \text{--- (i)}$$

For $f(x) = x$

$$w_1 x_1 + w_2 x_2 = \int_{-1}^1 x \cdot dx = \frac{x^2}{2} \Big|_{-1}^1 = \frac{(1)^2}{2} - \frac{(-1)^2}{2}$$

$$= \frac{1}{2} - \frac{1}{2} = 0 \quad \text{--- (ii)}$$

For $f(x) = x^2$

$$w_1 x_1^2 + w_2 x_2^2 = \int_{-1}^1 x^2 \cdot dx = \frac{x^3}{3} \Big|_{-1}^1 = \frac{1}{3} - \frac{(-1)^3}{3} = \frac{2}{3} \quad \text{--- (iii)}$$

For $f(x) = x^3$

$$w_1 x_1^3 + w_2 x_2^3 = \int_{-1}^1 x^3 dx = 0 \quad \text{--- (10)}$$

Solving eqⁿ (i), (ii), (iii) & (iv) we get

$$w_1 = w_2 = 1, \quad x_1 = -\frac{1}{\sqrt{3}}, \quad x_2 = +\frac{1}{\sqrt{3}}$$

$$\begin{aligned} \therefore I_g &= \int_{-1}^1 f(x) \cdot dx = w_1 f(x_1) + w_2 f(x_2) \\ &= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \end{aligned}$$

Q Evaluate $\int_{-1}^1 e^x \cdot dx$ by Gauss two point method

Solⁿ Here,

$$f(x) = e^x$$

Using Gauss Integration,

$$\begin{aligned} I &= \int_{-1}^1 e^x \cdot dx = w_1 f(x_1) + w_2 f(x_2) \\ &= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \end{aligned}$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = e^{-1/\sqrt{3}} = 0.5614$$

$$f\left(\frac{1}{\sqrt{3}}\right) = e^{1/\sqrt{3}} = 1.7813$$

$$= 0.5614 + 1.7813$$

$$\therefore I = 2.3427$$

$$x = Az + B$$

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Changing the limit of integration

$$\text{let, } \int_a^b f(x) dx = c \cdot \int_{-1}^1 g(z) dz \quad \text{--- (I)}$$

Assuming the following transformation between x & z ,

$$x = Az + B \quad \text{--- (II)}$$

Then it must satisfy the condition:

(comparing on eqⁿ (I) with) $\rightarrow$ when $x = a \Rightarrow z = -1 \therefore a = -A + B \quad \text{--- (III)}$

" $x = b \Rightarrow z = 1 \therefore b = A + B \quad \text{--- (IV)}$

Solving eqⁿ (III) and (IV), we get, for A and B

$$\therefore A = \frac{b-a}{2}$$

$$\therefore B = \frac{a+b}{2}$$

substituting the values of A and B in eqⁿ (II)

$$\therefore x = \frac{b-a}{2} z + \frac{a+b}{2} \quad \left\{ \frac{dx}{dz} = \frac{b-a}{2} \cdot 1 + 0 \right\}$$

$$dx = \frac{b-a}{2} dz \quad \Rightarrow dx = \frac{b-a}{2} dz$$

$$\therefore \int_a^b f(x) \cdot dx = \int_{-1}^1 g(z) \cdot \frac{b-a}{2} \cdot dz$$

$$\therefore \int_a^b f(x) \cdot dx = \frac{b-a}{2} \cdot \int_{-1}^1 g(z) \cdot dz = \frac{b-a}{2} \{g(-1/\sqrt{3}) + g(1/\sqrt{3})\}$$

Q

$$\int_{-2}^2 e^{-x/2} \cdot dx$$

Here,

Solⁿ

$$f(x) = e^{-x/2}$$

We have,

The transformation eqⁿ:

$$x = \frac{b-a}{2} z + \frac{b+a}{2}$$

$$\therefore x = 2z$$

$$\therefore g(z) = e^{-2z/2} = e^{-z}$$

we know,

$$I = \frac{b-a}{2} \left(g(-1/\sqrt{3}) + g(1/\sqrt{3}) \right)$$

$$= 2 \left(e^{1/\sqrt{3}} + e^{-1/\sqrt{3}} \right)$$

$$= 2 (1.7813 + 0.5614)$$

$$= 2 \times 2.3427$$

$$= 4.6854 \text{ Q}$$

Gauss three-point formula: $I = w_1 \cdot f(x_1) + w_2 \cdot f(x_2) + w_3 \cdot f(x_3)$.

Gauss integration assumes an approximation of the form

$$I_g = \int_{-1}^1 f(x) \cdot dx = \sum_{i=1}^n w_i \cdot f(x_i)$$

$$= w_1 \cdot f(x_1) + w_2 \cdot f(x_2) + w_3 \cdot f(x_3)$$

[For $n=3$]

And, the values of $w_1, w_2, w_3, x_1, x_2, x_3$ needs to be calculated.

For $f(x) = 1$ [$f(x_1) = f(x_2) = f(x_3) = 1$],

$$w_1 + w_2 + w_3 = \int_{-1}^1 1 \cdot dx$$

$$= x \Big|_{-1}^1$$

$$= 2 \quad \text{--- (i)}$$

For $f(x) = x$,

$$w_1 x_1 + w_2 x_2 + w_3 x_3 = 0 \quad \text{--- (ii)}$$

For $f(x) = x^2$,

$$w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 = \frac{2}{3} \quad \text{--- (iii)}$$

For $f(x) = x^3$,

$$w_1 x_1^3 + w_2 x_2^3 + w_3 x_3^3 = 0 \quad \text{--- (10)}$$

Solving eqⁿ (i), (ii), (iii) & (iv), we get

$$w_1 = 0.55556$$

$$x_1 = -0.77460$$

$$w_2 = 0.88889$$

$$x_2 = 0.00000$$

$$w_3 = 0.55556$$

$$x_3 = +0.77460$$

$$\therefore x' = \frac{(b-a)}{2} z + \frac{(b+a)}{2}$$

which is Gauss three-point formula.

Romberg Integration.• Steps

- i) Compute the integration with 'h' by trapezoidal rule.
- ii) Each time half the interval i.e. 'h/2' and compute the integral using trapezoidal rule.
- iii) Refine with Romberg integration formula

$$I = I_2 + \frac{1}{3} (I_2 - I_1)$$

Q $I = \int_0^1 \frac{dx}{1+x^2}$ for $h = 0.5, 0.25, 0.125$.

Sol $f(x) = \frac{1}{1+x^2}$

② For $h = 0.5$,

x	0	0.5	1
f(x)	1	0.8	0.5

$$I_1 = \frac{h}{2} [1 + 0.5 + 2 \times 0.8]$$

$$= \frac{0.5 \times 3.1}{2} = 0.7750$$

For $h = 0.25$

X	0	0.25	0.5	0.75	1
f(x)	1	0.9412	0.8	0.64	0.5

$$I_2 = \frac{h}{2} [1 + 0.5 + 2 \times (0.9412 + 0.8 + 0.64)]$$

$$= \frac{0.25}{2} \times 6.2624$$

$$= 0.7833$$

For $h = 0.125$

X	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875
f(x)	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664

X	1
f(x)	0.5

$$I_3 = \frac{h}{2} [1 + 0.5 + 2 \times (0.9846 + 0.9412 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.5664)]$$

$$= 0.7847$$

$$I_1 = 0.7750$$

$$I_2 = 0.7833$$

$$I_3 = 0.7847$$

$$I' = I_2 + \frac{1}{3}(I_2 - I_1) = 0.7861$$

$$I'' = I_3 + \frac{1}{3}(I_3 - I_2) = 0.7852$$

$$\begin{aligned} I''' &= I'' + \frac{1}{3}(I'' - I') \\ &= 0.7849 \end{aligned}$$

(5)

Numerical Differentiation

- For equally spaced data

↳ Newton's forward interpolation method
 ↳ " backward " "

a) Using Newton's forward interpolation formula.

The Newton's forward interpolation formula is given by

$$f(x) = f_0 + s \Delta f_0 + \frac{s(s-1)}{2!} \Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!} \Delta^3 f_0 + \frac{s(s-1)(s-2)(s-3)}{4!} \Delta^4 f_0 + \dots$$

where,

$$x = x_0 + sh$$

$$\text{or, } dx = h \cdot ds$$

$$\text{or, } \frac{ds}{dx} = \frac{1}{h} \quad \text{--- (i)}$$

Now, we know,

$$\frac{df}{dx} = \frac{df}{ds} \cdot \frac{ds}{dx} = \frac{1}{h} \left[0 + \Delta f_0 + \frac{2s-1}{2!} \Delta^2 f_0 + \frac{3s^2-6s+2}{3!} \Delta^3 f_0 + \dots \right] \quad \text{--- (ii)}$$

Now, to find second derivative we have.

$$\begin{aligned}
 \frac{d^2 f}{dx^2} &= \frac{d}{ds} \cdot \left(\frac{df}{dx} \right) \cdot \frac{ds}{dx} \\
 &= \frac{1}{h} \cdot \frac{d}{ds} \left[\frac{1}{h} \left(\Delta f_0 + \frac{2s-1}{2!} \Delta^2 f_0 + \frac{3s^2-6s+2}{3!} \Delta^3 f_0 + \dots \right) \right] \\
 &= \frac{1}{h^2} \left[\Delta^2 f_0 + \frac{6s-6}{3!} \Delta^3 f_0 + \dots \right] \\
 &= \frac{1}{h^2} \left[\Delta^2 f_0 + (s-1) \Delta^3 f_0 + \dots \right] \quad \text{--- (iv)}
 \end{aligned}$$

• Special case

If the derivative we want to find is at the beginning point i.e. $x = x_0$ then

$x = x_0 + sh$ gives $s = 0$ then eqⁿ (iii) and (iv) becomes

$$\left(\frac{df}{dx} \right)_{x=x_0} = \frac{1}{h} \left[\Delta f_0 - \frac{1}{2} \Delta^2 f_0 + \frac{1}{3} \Delta^3 f_0 - \frac{1}{4} \Delta^4 f_0 + \dots \right]$$

$$\left(\frac{d^2 f}{dx^2} \right)_{x=x_0} = \frac{1}{h^2} \left[\Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12} \Delta^4 f_0 + \dots \right]$$

Using Newton's Backward Interpolation formula

The Newton's Backward interpolation formula is given as:

$$f(x) = f_n + s \Delta f_n + \frac{s(s+1)}{2!} \Delta^2 f_n + \frac{s(s+1)(s+2)}{3!} \Delta^3 f_n + \frac{s(s+1)(s+2)(s+3)}{4!} \Delta^4 f_n + \dots \quad \text{--- (a)}$$

Where, $x = x_n + sh$

or, $dx = h \cdot ds$

$$\therefore \frac{ds}{dx} = \frac{1}{h}$$

Now, we know,

$$\frac{df}{dx} = \frac{df}{ds} \cdot \frac{ds}{dx} = \frac{1}{h} \left[0 + \Delta f_n + \frac{s+1}{2!} \Delta^2 f_n + \frac{(3s^2+6s+2)}{3!} \Delta^3 f_n + \dots \right] \quad \text{--- (b)}$$

Now, to find second derivative we have

$$\frac{d^2f}{dx^2} = \frac{d}{ds} \left(\frac{df}{dx} \right) \cdot \frac{ds}{dx}$$

$$= \frac{1}{h} \cdot \frac{d}{ds} \left[\frac{1}{h} \left(\Delta f_n + \frac{(s+1)}{2!} \Delta^2 f_n + \frac{(3s^2+6s+2)}{3!} \Delta^3 f_n + \dots \right) \right]$$

$$= \frac{1}{h^2} [\Delta^2 f_n + (s+1) \Delta^3 f_n + \dots] \quad \text{--- (c)}$$

• Special case

If the derivate we want to find is at the end point i.e. $x = x_n$ then,

$x = x_n + sh$ gives $s=0$ then eqⁿ (b) and (c) becomes,

$$\left(\frac{df}{dx} \right)_{x=x_n} = \frac{1}{h} \left[\Delta f_n + \frac{1}{2} \Delta^2 f_n + \frac{1}{3} \Delta^3 f_n + \frac{1}{4} \Delta^4 f_n + \dots \right]$$

$$\left(\frac{d^2 f}{dx^2} \right)_{x=x_n} = \frac{1}{h^2} \left[\Delta^2 f_n + \Delta^3 f_n + \frac{1}{12} \Delta^4 f_n + \dots \right]$$

Q

Given table below

x	1	2	3	4	5	6	7
f(x)	0.000	0.693	1.099	1.386	1.609	1.792	1.946

Obtain the value of $f'(1)$, $f'(2)$, $f'(7)$ and $f''(1)$

$$h = 2 - 1 = 1$$

Solⁿ

Constructing divided difference table, we have

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
1	0.000	0.693	-0.2861	0.1662	-0.1103	0.0784	-0.0538
2	0.693	0.4069	-0.1199	0.0559	-0.0319	0.0189	
3	1.099	0.287	-0.0640	0.0240	-0.0130		
4	1.386	0.223	-0.0400	0.0110			
5	1.609	0.183	-0.0290				
6	1.792	0.154					
7	1.946						

Now,

$$f'(1) = \left(\frac{df(x)}{dx} \right)_{x=x_0} = \frac{1}{h} \left[\Delta f_0 - \frac{1}{2} \Delta^2 f_0 + \frac{1}{3} \Delta^3 f_0 - \frac{1}{4} \Delta^4 f_0 + \frac{1}{5} \Delta^5 f_0 - \frac{1}{6} \Delta^6 f_0 \right]$$

$$= \frac{1}{1} \left[0.693 - \frac{1}{2} \times (-0.2861) + \frac{1}{3} \times 0.1662 \right]$$

$$\therefore S = \frac{x - x_n}{h}$$

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$$-\frac{1}{4} \times (-0.1103) + \frac{1}{5} \times 0.0784 - \frac{1}{6} \times (-0.0595)]$$

$$= 1.0609 \%$$

$$\therefore F''(1) = \frac{1}{h^2} [\Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12} \Delta^4 f_0]$$

$$= [-0.2861 - 0.2870 + \frac{11}{12} \times (-0.1103)]$$

$$= -0.7867 \%$$

$$\therefore F'(7) = \left(\frac{dF(x)}{dx} \right)_{x=x_n} = \frac{1}{h} [\Delta f_n + \frac{1}{2} \Delta^2 f_n + \frac{1}{3} \Delta^3 f_n +$$

$$\frac{1}{4} \Delta^4 f_n + \frac{1}{5} \Delta^5 f_n + \frac{1}{6} \Delta^6 f_n]$$

$$= 0.1130 \%$$

Best
Process

$$F'(2) = \frac{1}{h} [\Delta f_0 + \frac{2S+1}{2!} \Delta^2 f_0 + \frac{3S^2+6S+2}{3!} \Delta^3 f_0 + \frac{4S^3+18S^2+28S+6}{4!}$$

$$\frac{\Delta^4 f_0 + 5S^4 + 40S^3 + 105S^2 - 20S + 24}{5!} \Delta^5 f_0 + \frac{6S^5 - 76S^4 + 340S^3 - 51S^2 + 148S - 12}{6!} \Delta^6 f_0]$$

$$= 0.6434 \%$$

#

Numerical differentiation of equally spaced data from Taylor's series :

According to Taylor's series theorem,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) \quad \text{for } x \leq a < x+h$$

$$\text{or, } f'(x) = \frac{1}{h} f(x+h) - \frac{h}{2} f''(x)$$

If h is chosen too small then higher values can be neglected,

$$\therefore f'(x) = \frac{f(x+h) - f(x)}{h} \quad \text{--- (I)}$$

$$\text{truncation error } E_t(h) = -\frac{h}{2} f''(x)$$

Here, eqn (I) is known as first order forward difference quotient and is also called two point formula.

Similarly,

$$f'(x) = \frac{f(x) - f(x-h)}{h} \quad \text{--- (II)}$$

is known as backward difference quotient.

Q If $f(x) = x^2$, Find $f'(x)$ at $x=1$ with $h = 0.2, 0.1, 0.05$.

Solⁿ For $h = 0.2, x = 1$

$$\text{Forward} \rightarrow f'(x) = \frac{f(x+h) - f(x)}{h} = \frac{f(1+0.2) - f(1)}{0.2} = \frac{(1.2)^2 - (1)^2}{0.2} = 2.2$$

$$\text{Backward} \rightarrow f'(x) = \frac{f(x) - f(x-h)}{h} = \frac{f(1) - f(1-0.2)}{0.2} = \frac{(1)^2 - (0.8)^2}{0.2} = 1.8$$

For $h = 0.1, x = 1$

$$\text{Forward} \rightarrow f'(x) = \frac{f(1+0.1) - f(1)}{0.1} = \frac{(1.1)^2 - (1)^2}{0.1} = 2.1$$

$$\text{Backward} \rightarrow f'(x) = \frac{f(1) - f(1-0.1)}{0.1} = \frac{(1)^2 - (0.9)^2}{0.1} = 1.9$$

For $h = 0.05, x = 1$

$$\text{Forward} \rightarrow f'(x) = \frac{f(1+0.05) - f(1)}{0.05} = \frac{(1.05)^2 - (1)^2}{0.05} = 2.05$$

$$\text{Backward} \rightarrow f'(x) = \frac{f(1) - f(1-0.05)}{0.05} = \frac{(1)^2 - (0.95)^2}{0.05} = 1.95$$

central difference quotient

As,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f'''(x_1) \quad \text{--- (A)}$$

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{3!} f'''(x_2) \quad \text{--- (B)}$$

Subtracting (B) from (A)

$$f(x+h) - f(x-h) = 2h f'(x) + \frac{h^3}{3!} [f'''(x_1) + f'''(x_2)]$$

neglecting higher terms,

$$f(x+h) - f(x-h) = 2h f'(x)$$

$$\therefore f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

which is known as central difference quotient.

Q

$$f(x) = x^2 \text{ at } x=1, h=0.2, 0.1, 0.05.$$

Solⁿ

$$\text{For } h=0.2, x=1$$

$$f'(x) = \frac{f(1+0.2) - f(1-0.2)}{2 \times 0.2} = \frac{(1.2)^2 - (0.8)^2}{0.4} = ?$$

For $h = 0.1, x = 1$

$$f'(x) = \frac{f(1+0.1) - f(1-0.1)}{2 \times 0.1} = \frac{(1.1)^2 - (0.9)^2}{0.2} = 2$$

For $h = 0.05, x = 1$

$$f'(x) = \frac{f(1+0.05) - f(1-0.05)}{2 \times 0.05} = \frac{(1.05)^2 - (0.95)^2}{0.1} = 2$$

Maximum & minimum of tabulated function.

→ Maximum and minimum values of a function can be found by equating the first derivative to zero and solving for the variable.

→ Same procedure is employed for tabulated function.

Now, according to Newton's forward difference formula:

$$f(s) = f_0 + s \Delta f_0 + \frac{s(s-1)}{2!} \Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!} \Delta^3 f_0 + \dots \quad \text{--- (1)}$$

Differentiating eqⁿ (1) w.r.t. 's'.

$$f'(s) = \Delta f_0 + \frac{(2s-1)}{2!} \Delta^2 f_0 + \frac{(3s^2-6s+2)}{3!} \Delta^3 f_0 + \dots \quad \text{--- (1)}$$

For maxima or minima, we have $f'(s) = 0$

Hence, neglecting higher terms eqⁿ (1) can be expressed as

$$f'(s) = 0$$

$$C_0 + C_1 s + C_2 s^2 = 0$$

where, $C_0 = \Delta f_0 - \frac{1}{2} \Delta^2 f_0 + \frac{1}{3} \Delta^3 f_0$

$$C_1 = \Delta^2 f_0 - \Delta^3 f_0$$

$$C_2 = \frac{1}{2} \Delta^3 f_0$$

$$\text{and } x = x_0 + sh$$

Q From the following table, find x correct to two decimal places for which y is maximum and find the value of y

x	1.2	1.3	1.4	1.5	1.6
y	0.9320	0.9636	0.9855	0.9975	0.9996

1.5758



Solⁿ

$$h = 1.3 - 1.2 = 0.1$$

Constructing difference table,

→ correct to
↓ two decimal
place

Use Newton's,

X	y	Δy	$\Delta^2 y$
1.2	0.9320	0.0316	-0.0087
1.3	0.9636	0.0219	-0.0099
1.4	0.9855	0.0120	-0.0099
1.5	0.9975	0.0021	
1.6	0.9996		

We have,

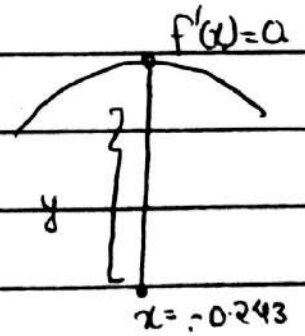
For x ,

$$f'(s) = \Delta f_0 + (2s-1) \frac{\Delta^2 f_0}{2!}$$

$$0 = 0.0316 + 2s-1 \cdot (-0.0087)$$

$$\therefore s_1 = 3.7576$$

$$\therefore x = x_0 + s_1 h = 1.5758$$



Next,

$$x = 1.5758, s_2 = \frac{x - x_n}{h} = \frac{1.5758 - 1.6}{0.1} = -0.243$$

For y ,

Applying Newton's backward interpolation formula,

$$f_n(s) = \Delta f_n + s_2 \Delta f_n + \frac{s_2^2}{2!} (s_2+1) \Delta^2 f_n$$

$$\text{or, } y(1.58) = 0.9996 + (-0.243) \cdot (0.0021) + (-0.243) \cdot (-0.243 + 1) \cdot 2!$$

$$(0.0099) \\ = 0.9954 \quad \text{✓}$$

Solution of Linear Algebraic Equations

Any equation of linear type can be written as:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \text{--- ①}$$

Eqⁿ ① can be written in matrix form as

$$AX = B$$

$$\text{i.e. } [A]_{n \times n} [X]_{n \times 1} = [B]_{n \times 1}$$

Gauss Elimination Method

→ we make the augmented matrix an upper triangular matrix.

→ For first iteration, we apply $R_i = R_i - \frac{a_i}{a_{11}} R_1$ and so on.

Q Solve by Gauss Elimination method

$$3x + 2y + z = 10$$

$$2x + 3y + 2z = 11$$

$$x + 2y + 3z = 14$$

Solⁿ The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 10 \\ 2 & 3 & 2 & 11 \\ 1 & 2 & 3 & 14 \end{array} \right]$$

$$R_2 \rightarrow R_2 - \frac{2}{3} R_1$$

$$R_3 \rightarrow R_3 - \frac{1}{3} R_1$$

$$\sim \begin{bmatrix} 3 & 2 & 1 & : & 10 \\ 0 & 5/3 & 4/3 & : & 13/3 \\ 0 & 4/3 & 8/3 & : & 34/3 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 & : & 10 \\ 0 & 1.6667 & 1.3333 & : & 4.3333 \\ 0 & 1.3333 & 2.6667 & : & 10.667 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{1.3333}{1.6667} R_2$$

$$\sim \begin{bmatrix} 3 & 2 & 1 & : & 10 \\ 0 & 1.6667 & 1.3333 & : & 4.3333 \\ 0 & 0 & 1.6001 & : & 7.2002 \end{bmatrix}$$

Applying Backward Substitution,

$$1.6001 z = 7.2002$$

$$\therefore z = 4.4998$$

$$1.6667 y + 1.3333 z = 4.3333 \Rightarrow y = \frac{4.3333 - 1.3333 \times 4.4998}{1.6667}$$

$$\therefore y = -0.9998$$

$$3x + 2y + z = 10$$

$$\Rightarrow x = \frac{10 - 2 \times (-0.9998) - 4.4998}{3}$$

$$\therefore x = 2.4999$$

Gauss Elimination by Pivoting1) Partial Pivoting

- select the element with largest absolute value from first column and make it a pivot element by applying Row operation
- Continue the process for next iterations also and apply gauss elimination.

Q. Solve by Gauss Elimination with partial pivoting:

$$\begin{bmatrix} 1 & 1 & -2 \\ 4 & -2 & 1 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$$

Sol

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 4 & -2 & 1 & 5 \\ 3 & -1 & 3 & 8 \end{array} \right]$$

$$R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 4 & -2 & 1 & 5 \\ 1 & 1 & -2 & 3 \\ 3 & -1 & 3 & 8 \end{array} \right]$$

$$R_2 \rightarrow R_2 - \frac{1}{4} R_1$$

$$R_3 \rightarrow R_3 - \frac{3}{4} R_1$$

$$\sim \begin{bmatrix} 4 & -2 & 1 & : & 5 \\ 0 & 1.5 & -2.25 & : & 1.75 \\ 0 & 0.5 & 2.25 & : & 4.25 \end{bmatrix}$$

Since, pivot element in column 2 seems largest, so, interchange for partial pivoting is not required.

$$R_3 \rightarrow R_3 - \frac{0.5}{1.5} R_2$$

$$\sim \begin{bmatrix} 4 & -2 & 1 & : & 5 \\ 0 & 1.5 & -2.25 & : & 1.75 \\ 0 & 0 & 3 & : & 3.668 \end{bmatrix}$$

Applying Backward Substitution,

$$3x_3 = 3.668$$

$$\therefore x_3 = 1.2227$$

$$1.5x_2 - 2.25x_3 = 1.75$$

$$\therefore x_2 = \frac{1.75 + 2.25 \times 1.2227}{1.5} = 3.0007$$

$$4x_1 - 2x_2 + x_3 = 5$$

$$\therefore x_1 = \frac{5 - 1.2227 + 2 \times 3.0007}{4}$$

$$= 2.4447 \quad \%$$

ii) Complete pivoting

- We consider the entire coefficients and make the largest of them as pivot element and apply Gauss elimination.

Q

$$x_1 + 2x_2 + 3x_3 = 8$$

$$2x_1 + 4x_2 + 9x_3 = 8$$

$$4x_1 + 3x_2 + 2x_3 = 7$$

Soln

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & : & 8 \\ 2 & 4 & 9 & : & 8 \\ 4 & 3 & 2 & : & 7 \end{bmatrix}$$

$x_1 \quad x_2 \quad x_3$

Apply $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 2 & 4 & 9 & : & 8 \\ 1 & 2 & 3 & : & 8 \\ 4 & 3 & 2 & : & 7 \end{bmatrix}$$

$x_1 \quad x_2 \quad x_3$

Apply $C_3 \leftrightarrow C_1$

$$\sim \begin{bmatrix} 9 & 4 & 2 & : & 8 \\ 3 & 2 & 1 & : & 8 \\ 2 & 3 & 4 & : & 7 \end{bmatrix}$$

$x_3 \quad x_2 \quad x_1$

$$R_2 \rightarrow R_2 - \frac{3}{9} R_1$$

$$R_3 \rightarrow R_3 - \frac{2}{9} R_1$$

$$\sim \begin{bmatrix} 9 & 4 & 2 & : & 8 \\ 0 & 0.6667 & 0.3333 & : & 5.3333 \\ 0 & 2.1111 & 3.5556 & : & 0.2222 \end{bmatrix}$$

$x_3 \quad x_2 \quad x_1$

Since, 3.5556 is greatest among remaining pivot columns.

Placing it on second pivot column.

$$C_3 \leftrightarrow C_2$$

$$\sim \begin{bmatrix} 9 & 2 & 4 & : & 8 \\ 0 & 0.3333 & 0.6667 & : & 5.3333 \\ 0 & 3.5556 & 2.1111 & : & 0.2222 \end{bmatrix}$$

$x_3 \quad x_1 \quad x_2$

$$R_3 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 9 & 2 & 4 & : & 8 \\ 0 & 3.5556 & 2.1111 & : & 0.2222 \\ 0 & 0.3333 & 0.6667 & : & 5.3333 \end{bmatrix}$$

$x_3 \quad x_1 \quad x_2$

$$R_3 \rightarrow R_3 - \frac{0.3333}{3.5556} R_2$$

$$\sim \begin{bmatrix} 9 & 2 & 4 & : & 8 \\ 0 & 3.5556 & 2.1111 & : & 0.2222 \\ 0 & 0 & 0.4688 & : & 5.3125 \end{bmatrix}$$

$x_3 \qquad x_1 \qquad x_2$

Applying backward substitution

$$0.4688 x_2 = 5.3125$$

$$\therefore x_2 = 11.3321$$

$$3.5556 x_1 + 2.1111 x_2 = 0.2222$$

$$\therefore x_1 = -6.6658$$

$$9x_3 + 2x_1 + 4x_2 = 8$$

$$\therefore x_3 = -2.6663$$

Gauss - Jordan Method.

The coefficient of matrix A of $AX=B$ is reduced to identity matrix.

The result of Gauss-Jordan method is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Q. Solve :

$$10x + y + z = 12$$

$$2x + 10y + z = 13$$

$$x + y + 5z = 7$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array} \right]$$

$$R_1 \rightarrow R_1/10$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0.1 & 0.1 & 1.2 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0.1 & 0.1 & : & 1.2 \\ 0 & 9.8 & 0.8 & : & 10.6 \\ 0 & 0.9 & 4.9 & : & 5.8 \end{bmatrix}$$

$$R_2 \rightarrow R_2 / 9.8$$

$$\sim \begin{bmatrix} 1 & 0.1 & 0.1 & : & 1.2 \\ 0 & 1 & 0.08163 & : & 1.08163 \\ 0 & 0.9 & 4.9 & : & 5.8 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 0.9 R_2$$

$$\sim \begin{bmatrix} 1 & 0.1 & 0.1 & : & 1.2 \\ 0 & 1 & 0.08163 & : & 1.08163 \\ 0 & 0 & 4.8265 & : & 4.8265 \end{bmatrix}$$

$$R_3 \rightarrow R_3 / 4.8265$$

$$\sim \begin{bmatrix} 1 & 0.1 & 0.1 & : & 1.2 \\ 0 & 1 & 0.08163 & : & 1.08163 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 0.08163 R_3$$

$$R_1 \rightarrow R_1 - 0.1 R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 1.1 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - 0.1 R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$$\therefore x = 1, y = 1, z = 1 \quad \text{Q}$$

Inverse of a matrix by Gauss Jordan Method

Let A^{-1} be the inverse of A then we must have,

$$A A^{-1} = I$$

→ To find the inverse by Gauss-Jordan method we start with $[A : I]$ and apply Gauss-Jordan method until we get $[I : B]$. Here, B is the inverse matrix of A i.e. $B = A^{-1}$.

Q Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} = A$

Solⁿ For inverse of A ,

$$[A : I]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 5 & 0 & 1 & 0 \\ 2 & 5 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 5 & 0 & 1 & 0 \\ 0 & 3 & -3 & -2 & 0 & 1 \end{array} \right]$$

$$R_2 \rightarrow R_2/2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & 1 & 2.5 & : & 0 & 0.5 & 0 \\ 0 & 3 & -3 & : & -2 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & 1 & 2.5 & : & 0 & 0.5 & 0 \\ 0 & 0 & -10.5 & : & -2 & -1.5 & 1 \end{bmatrix}$$

$$R_3 \rightarrow -R_3 / 10.5$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & 1 & 2.5 & : & 0 & 0.5 & 0 \\ 0 & 0 & 1 & : & 0.1905 & 0.1429 & -0.0952 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2.5R_3$$

$$R_1 \rightarrow R_1 - R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & : & 0.8095 & -0.1429 & 0.0952 \\ 0 & 1 & 0 & : & -0.4763 & 0.1428 & 0.2380 \\ 0 & 0 & 1 & : & 0.1905 & 0.1429 & -0.0952 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 1.2858 & -0.2857 & -0.1428 \\ 0 & 1 & 0 & : & -0.4763 & 0.1428 & 0.2380 \\ 0 & 0 & 1 & : & 0.1905 & 0.1429 & -0.0952 \end{bmatrix}$$

$$\therefore \text{Inverse} = \begin{bmatrix} 1.2858 & -0.2857 & -0.1428 \\ -0.4763 & 0.1428 & 0.2380 \\ 0.1905 & 0.1429 & -0.0952 \end{bmatrix}$$

Triangular Factorization method / LU decomposition method

The system of linear eqⁿ is given by

$$AX = B \quad \text{--- (1)}$$

The triangular factorization method decomposes the coefficient matrix 'A' into two triangular matrices L and U given as

$$A = LU \quad \text{--- (2)}$$

Then $AX = B$ becomes

$$(LU)X = B$$

$$\text{or, } LUX = B$$

$$\text{or, } LY = B \quad \text{where, } UX = Y$$

where,

L is the lower triangular matrix of the form

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \dots & l_{nn} \end{bmatrix}$$

U is the upper triangular matrix of the form

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

So, the problem can be solved in two steps:

- Finding y from $Ly=B$ by forward substitution where $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$
- Solving $Ux=y$ for x by backward substitution.

→ If $a_{11} = a_{22} = \dots = a_{nn} = 1$ in the matrix L then the method is called Dolite Decomposition.

Q Solve by Dolite Decomposition.

$$2x + 3y + z = 9$$

$$x + 2y + 3z = 6$$

$$3x + y + 2z = 8$$

Solⁿ

Representing the above eqⁿ in matrix form and comparing with $AX=B$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

where, $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

We Factorize A such that,

$$A = LU$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\text{ex, } \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21} U_{11} & L_{21} U_{12} + U_{22} & L_{21} U_{13} + U_{23} \\ L_{31} U_{11} & L_{31} U_{12} + L_{32} U_{22} & L_{31} U_{13} + L_{32} U_{23} + U_{33} \end{bmatrix}$$

Comparing the coefficient, we get

$$\therefore U_{11} = 2$$

$$\therefore U_{12} = 3$$

$$\therefore U_{13} = 1$$

$$\Rightarrow L_{21} = \frac{1}{2} = 0.5$$

$$\therefore U_{22} = 2 - (0.5 \times 3) = 0.5$$

$$\therefore U_{23} = 3 - (0.5 \times 1) = 2.5$$

$$\Rightarrow L_{31} = 3/2 = 1.5$$

$$\therefore L_{32} = \frac{1 - (1.5 \times 3)}{0.5} = -7$$

$$\therefore U_{33} = 2 - (1.5 \times 1) - (-7 \times 2.5) = 18$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -7 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0.5 & 2.5 \\ 0 & 0 & 18 \end{bmatrix}$$

Now, For $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ we have

$$LY = B$$

$$\text{ex, } \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$$1y_1 = 9 \Rightarrow y_1 = 9$$

$$0.5y_1 + y_2 = 6 \Rightarrow y_2 = 6 - (0.5 \times 9) = 1.5$$

$$1.5y_1 - 7y_2 + y_3 = 8 \Rightarrow y_3 = 8 - (1.5 \times 9) + (7 \times 1.5) = 5$$

Now, we have,

$$\text{or, } UX = Y$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 0.5 & 2.5 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1.5 \\ 5 \end{bmatrix}$$

$$18z = 5 \Rightarrow z = 5/18 = 0.2778$$

$$0.5y + 2.5z = 1.5 \Rightarrow y = \frac{1.5 - (2.5 \times 0.2778)}{0.5} = 1.6110$$

$$2x + 3y + z = 9 \Rightarrow x = \frac{9 - 0.2778 - 3 \times 1.6110}{2} = 1.9446$$

Cholesky Factorization

In this method, we take A as

$$A = U^T U \quad \text{or} \quad LL^T.$$

Factorize the matrix using Cholesky factorization

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = A$$

Let, $U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$ and $U^T = \begin{bmatrix} U_{11} & 0 & 0 \\ U_{12} & U_{22} & 0 \\ U_{13} & U_{23} & U_{33} \end{bmatrix}$

Using Cholesky Factorization,



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$$A = U^T U$$

$$\text{eg, } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} U_{11} & 0 & 0 \\ U_{12} & U_{22} & 0 \\ U_{13} & U_{23} & U_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\text{eg, } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} U_{11}^2 & U_{11} U_{12} & U_{11} U_{13} \\ U_{12} U_{11} & U_{12}^2 + U_{22}^2 & U_{12} U_{13} + U_{22} U_{23} \\ U_{13} U_{11} & U_{13} U_{12} + U_{23} U_{22} & U_{13}^2 + U_{23}^2 + U_{33}^2 \end{bmatrix}$$

Comparing the coefficient, we get

$$\therefore U_{11} = 1$$

$$\therefore U_{12} = 2/1 = 2$$

$$\therefore U_{13} = 3/1 = 3$$

$$\Rightarrow U_{12}^2 + U_{22}^2 = 8$$

$$\text{eg, } U_{22}^2 = 8 - (2)^2 = 4$$

$$\therefore U_{22} = 2$$

$$\Rightarrow U_{23} = \frac{-U_{12} \cdot U_{13} + 22}{U_{22}}$$

$$= \frac{-2 \times 3 + 22}{2}$$

$$= 8$$

$$\Rightarrow U_{13}^2 + U_{23}^2 + U_{33}^2 = 82$$

$$U_{33}^2 = 82 - U_{13}^2 - U_{23}^2$$

$$= 82 - (3)^2 - (8)^2$$

$$= 9$$

$$\therefore U_{33} = 3$$

$$\therefore U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 8 \end{bmatrix} \text{ and } U^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 8 \end{bmatrix}$$

Iterative methods

i) Gauss Jacobi method

ii) Gauss Seidel method

Let, the set of eq's be

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

From above eq's, we can write

$$x_1 = \frac{b_1 - a_{12}x_2 - \dots - a_{1n}x_n}{a_{11}} \quad \text{--- (i)}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - \dots - a_{2n}x_n}{a_{22}} \quad \text{--- (ii)}$$

⋮

⋮

⋮

$$x_n = \frac{b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{nn}x_{n-1}}{a_{nn}} \quad \text{--- (n)}$$

Using these 'n' equations, we can have the iterative solⁿ by Gauss Jacobi and Gauss Seidel method with initial guess $x_1 = x_2 = x_3 = \dots = x_n = x_0$

- In ^{Gauss} Jacobi method, take the values obtained from previous iteration.
- In Gauss Seidel method use the most latest value available for calculation.

Q Solve by Gauss Jacobi and Gauss Seidel method.

$$2x - 3y + 20z = 25$$

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

Solⁿ From above eqn^s, we can write

$$x = \frac{25 + 3y - 20z}{2} \quad \text{--- (i)}$$

$$y = 17 - 20x + 2z \quad \text{--- (ii)}$$

$$z = 18 + 3x + 20y \quad \text{--- (iii)}$$

(initial guess)

let $x=y=z=0$ then new values are

$$x = \frac{25 + 0 - 0}{2} = 12.5$$

$$y = 17$$

$$z = 18$$

n	x	y	z
0	0	0	0
1	12.5	17	18
2	-142	-197	369.5

Process Initial n
0
1



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3	-4238	3312	-4348
4	48964.5	76081	60284

It doesn't converge.

$$2x - 3y + 20z = 25 \Rightarrow z = \frac{25 - 2x + 3y}{20}$$

$$20x + y - 2z = 17 \Rightarrow x = \frac{17 - y + 2z}{20}$$

$$3x + 20y - z = -18 \Rightarrow y = \frac{-18 - 3x + z}{20}$$

let, $x = y = z = 0$

$$\therefore z = \frac{25 + 0 + 0}{20} = 1.25$$

$$\therefore x = \frac{17 - 0 + 0}{20} = 0.85$$

$$\therefore y = \frac{-18 - 0 + 0}{20} = -0.9$$

Gauss-Jacobi method

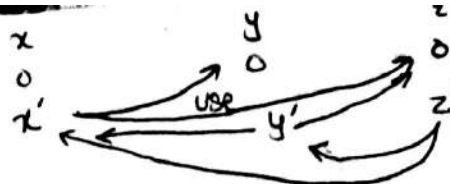
n	x	y	z
0	0	0	0
1	0.85	-0.9	1.25
2	1.02	-0.965	1.03
3	1.0012	-1.0015	1.00325
4	1.0004	-1.0008	0.99985
5	0.9999	-1.0000	0.9997
6	0.9999	-1.0000	1.0000

$$\therefore x = 0.9999 \approx 1, y = -1, z = 1$$

Process

Initial

n
0
1



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Gauss Seidel method

n	x'	y	z
0	0	0	0
1	0.85	-1.027	1.01035
2	1.0024	-0.9998	0.9997
3	0.9999	-1	1.0000
4	1	-1	1
5	1	-1	1

$\therefore x=1, y=-1$ and $z=1$ $\Rightarrow$

Eigen value and Eigen vectors

let, us consider a square matrix A of size $n \times n$ and a non-zero column vector X such that

$$AX = \lambda X \quad \text{--- (I)}$$

where, λ is a scalar quantity and is called eigen value or latent root or characteristic value of A .

Eq (I) can be written as

$$AX - \lambda X = 0$$

$$\text{or } (A - \lambda I)X = 0 \quad \text{--- (II)}$$

$$\text{or, } \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ 0 \end{bmatrix} = 0 \quad \text{--- (III)}$$

Eq (II) & (III) represent a set of linear homogeneous eqⁿ. The necessary condition for its solution to exist is

$$|A - \lambda I| = 0$$

and this eqⁿ is called characteristic eqⁿ of A . The roots of characteristic eqⁿ is known as Eigen values ($\lambda_1, \lambda_2, \dots, \lambda_n$).

The corresponding Eigen vectors are

$$AX_1 = \lambda_1 X_1, \quad AX_2 = \lambda_2 X_2, \quad AX_3 = \lambda_3 X_3, \quad \dots \quad AX_n = \lambda_n X_n$$

Q

Find the Eigen value & Eigen vector of matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix} = A(\text{let})$$

SolⁿThe characteristic eqⁿ is

$$|A - \lambda I| = 0$$

$$\text{or, } \left| \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\text{or, } \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & 1 \\ 5 & 4 & 1-\lambda \end{vmatrix} = 0$$

Expanding along R_1 , we get

$$(1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 5 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 2 & 1-\lambda \\ 5 & 4 \end{vmatrix} = 0$$

$$\text{or, } (1-\lambda) (1-2\lambda+\lambda^2-4) - 2 (2-2\lambda-5) - 1 (8-5+5\lambda) = 0$$

$$\text{or, } (1-\lambda) (\lambda^2-2\lambda-3) - 2 (-2\lambda-3) - 1 (5\lambda+3) = 0$$

$$\text{or, } (1-\lambda) (\lambda^2-2\lambda-3) + 4\lambda+6-5\lambda-3 = 0$$

$$\text{or, } (1-\lambda) (\lambda^2-2\lambda-3) - \lambda+3 = 0$$

$$\text{or, } \lambda^2-2\lambda-3 - \lambda^3+2\lambda^2+3\lambda-\lambda+3 = 0$$

$$\text{or, } -\lambda^3+3\lambda^2 = 0$$

$$\text{or, } \lambda^2 (3-\lambda) = 0$$

Either,

$$\lambda^2 = 0$$

$$\therefore \lambda = 0, 0$$

Or,

$$3 - \lambda = 0$$

$$\therefore \lambda = 3$$

$$\therefore \lambda = 0, 0, 3$$

For $\lambda_1 = \lambda_2 = 0$ The Eigen vector is given by eqⁿ

$$A X = \lambda_1 X$$

$$\text{or, } AX = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Applying Gauss Elimination.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 5 & 4 & 1 & 0 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 5R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \end{array} \right] \begin{array}{l} x_1 + 2x_2 - x_3 = 0 \text{ --- (i)} \\ -3x_2 + 3x_3 = 0 \text{ --- (ii)} \\ -6x_2 + 6x_3 = 0 \text{ --- (iii)} \end{array}$$

Since eqⁿ (i) & (iii) are same, hence it doesn't have unique solⁿ.

$$\text{let, } x_3 = k$$

$$-6x_2 + 6k = 0$$

$$\text{or, } x_2 = k \quad \text{From (1)}$$

$$x_2 = 3k/3 = k$$

From (1)

$$1 \cdot x_1 + 2x_2 - x_3 = 0$$

$$\text{or, } 1 \cdot x_1 + 2k - k = 0$$

$$\text{or, } x_1 = -k$$

$$\therefore \text{Eigenvector is } \begin{bmatrix} -k \\ k \\ k \end{bmatrix} = \begin{bmatrix} -k \\ k \\ k \end{bmatrix}$$

Again, For $\lambda_3 = 3$, The eigenvector is given by eqⁿ

$$Ax = \lambda_3 x$$

$$\text{or, } \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$(1-3)x_1 + 2x_2 - x_3 = 0$$

$$\Rightarrow -2x_1 + 2x_2 - x_3 = 0 \quad \text{--- (a)}$$

$$\text{or, } 2x_1 + (1-3)x_2 + x_3 = 0$$

$$2x_1 - 2x_2 + x_3 = 0 \quad \text{--- (b)}$$

$$\text{or, } 5x_1 + 4x_2 + (1-3)x_3 = 0$$

$$5x_1 + 4x_2 - 2x_3 = 0 \quad \text{--- (c)}$$

Since, eqⁿ (a) and (b) are same, hence it doesn't have unique solⁿ.

$$\text{let, } x_3 = k$$

From eqⁿ (a)

$$-2x_1 + 2x_2 = k \quad \text{--- (A)}$$

From eqⁿ (c)

$$5x_1 + 4x_2 = 2k \quad \text{--- (B)}$$

$$x_1 = 0, x_2 = \frac{k}{2}$$

∴ Eigen vector is $\begin{bmatrix} 0 \\ k/2 \\ k \end{bmatrix}$

∴ Eigen values are 0, 0, 1
and

Eigen vectors are $\begin{bmatrix} -k \\ k \\ k \end{bmatrix}$, $\begin{bmatrix} -k \\ k \\ k \end{bmatrix}$ and $\begin{bmatrix} 0 \\ k/2 \\ k \end{bmatrix}$ ↗

* Power Method

- Power method is an iterative method to find the dominant Eigen value.
- The Dominant Eigen value is the largest Eigen value in absolute magnitude.
- The corresponding Eigenvector is called dominant Eigen vector.

• Steps:

① let, $x_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

② Evaluate $z_i = Ax_i$

③ Evaluate $x_{i+1} = \frac{1}{c_{i+1}} z$

where c_{i+1} is the co-ordinate of z_i of largest magnitude.

④ Repeat ② & ③ till convergence.

① C_i and X_i gives λ and X respectively.

② Use power method to find the dominant eigen value and corresponding eigen vectors.

$$\begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix}$$

Solⁿ

$$\text{let, } A = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix}$$

①

$$\text{let, } X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

② Evaluate $Z_0 = AX_0 = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ 17 \end{bmatrix} = 18 \begin{bmatrix} 1 \\ 0.6667 \\ 0.9444 \end{bmatrix}$

$$Z_1 = AX_1 = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0.6667 \\ 0.9444 \end{bmatrix} = \begin{bmatrix} 17.2219 \\ 10.6112 \\ 14.1665 \end{bmatrix} = 17.2219 \begin{bmatrix} 1 \\ 0.6161 \\ 0.8229 \end{bmatrix}$$

$$Z_2 = AX_2 = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0.6161 \\ 0.8229 \end{bmatrix} = \begin{bmatrix} 16.1993 \\ 10.2873 \\ 12.7188 \end{bmatrix} = 16.1993 \begin{bmatrix} 1 \\ 0.6350 \\ 0.7951 \end{bmatrix}$$

$$Z_3 = AX_3 = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0.6350 \\ 0.7851 \end{bmatrix} = \begin{bmatrix} 15.9158 \\ 10.3251 \\ 15.2301 \end{bmatrix} = 15.9158 \begin{bmatrix} 1 \\ 0.6487 \\ 0.9569 \end{bmatrix}$$

$$Z_4 = AX_4 = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0.6487 \\ 0.9569 \end{bmatrix} = \begin{bmatrix} 17.3039 \\ 10.5517 \\ 14.1530 \end{bmatrix} = 17.3039 \begin{bmatrix} 1 \\ 0.6098 \\ 0.8179 \end{bmatrix}$$

$$Z_5 = AX_5 = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0.6098 \\ 0.8179 \end{bmatrix} = \begin{bmatrix} 16.1530 \\ 10.2571 \\ 12.6297 \end{bmatrix} = 16.1530 \begin{bmatrix} 1 \\ 0.6350 \\ 0.7851 \end{bmatrix}$$

Since, Z_2 and Z_5 have same eigen value and eigen vector.

$\therefore$ Dominant eigen value = 16.1530

" " Vector = $\begin{bmatrix} 1 \\ 0.6350 \\ 0.7851 \end{bmatrix}$

} 9

Numerical Solution of ordinary Differential Equations

$$\frac{dy}{dx} + 3y - 4 = 0 \quad \left[\begin{array}{l} \text{order} - 1 \\ \text{degree} - 1 \end{array} \right.$$

$$\left(\frac{d^3y}{dx^3}\right) + \left(\frac{dy}{dx}\right) + y = 0 \quad \begin{cases} \rightarrow \text{order} - 3 \\ \rightarrow \text{degree} - 1 \end{cases}$$

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^3 + y = 0 \quad \left[\begin{array}{l} \text{order} = 3 \\ \text{degree} = 3 \end{array} \right]$$

Solⁿ

→ Initial value problem → Initial values are given
→ Boundary " " → Boundary " " "

- Initial value problems:

① Taylor's series method

Tylox's series is given as:

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \dots$$

Representation:

$$\frac{dy}{dx} \rightarrow y' \quad \frac{d^2y}{dx^2} \rightarrow y''$$

$$\frac{dy^2}{dx} = \frac{dy^2}{dy} \cdot \frac{dy}{dx}$$

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Q

Find $y(x)$ for $y' = x - y^2$ and $y(0) = 1$ and $y(0.1) = ?$ Solⁿ

Here,

$$y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$$

$$y' = x - y^2 \Rightarrow y'(0) = x_0 - y_0^2 = (0) - (1)^2 = -1$$

$$y'' = 1 - 2yy' \Rightarrow y''(0) = 1 - 2y_0y'_0 = 1 - 2 \times 1 \times (-1) = 3$$

$$y''' = 0 - 2 \cdot \frac{d}{dx} (yy')$$

$$\Rightarrow y'''(0) = -2y_0y'_0 - 2(y'_0)^2$$

$$= -2 \times 1 \times 3 - 2 \times (-1)^2$$

$$= -6 - 2$$

$$= -8$$

$$= -2 \left[y \cdot \frac{dy'}{dx} + y' \cdot \frac{dy}{dx} \right]$$

$$= -2yy'' - 2(y')^2$$

Taylor's series is given as,

$$y(x) = 1 + (x-0)(-1) + \frac{(x-0)^2 \cdot 3}{2!} + \frac{(x-0)^3 \cdot (-8)}{3!} + \dots$$

$$= 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \dots$$

$$\therefore y(0.1) = 1 - 0.1 + \frac{3}{2} \times (0.1)^2 - \frac{4}{3} \times (0.1)^3 + \dots$$

$$\approx 0.9137$$

Euler's method (1st order RK method)

Considering only two terms of Taylor's series

$$y(x) = y(x_0) + y'(x_0)(x - x_0) \quad \text{--- (1)}$$

The given ODE is $y'(x) = F(x, y)$ with $y(x_0) = y_0$
 then $y'(x_0) = F(x_0, y_0)$

 $\therefore$ Eqⁿ (1) becomes,

$$y(x) = y(x_0) + F(x_0, y_0) \cdot (x - x_0)$$

Similarly,

$$y(x_1) = y(x_0) + F(x_0, y_0) \cdot (x_1 - x_0)$$

$$\text{i.e. } y_1 = y_0 + F(x_0, y_0) \cdot h$$

$$\therefore y_2 = y_1 + F(x_1, y_1) \cdot h$$

$$y_{i+1} = y_i + F(x_i, y_i) \cdot h$$

$$\text{Q. } y' = 3x^2 + 1 \text{ with } y(1) = 2$$

Estimate $y(2)$ using (i) $h = 0.5$ (ii) $h = 0.25$

Solⁿ

Here,

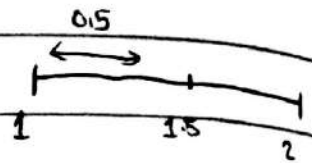
$$y(1) = 2 \Rightarrow x_0 = 1, y_0 = 2$$

$$F(x, y) = y' = 3x^2 + 1$$

Now,

Using Euler's formula,

$$y_{i+1} = y_i + F(x_i, y_i) \cdot h$$

①. For $h = 0.5$

$$y_1 = y(x_0 + h)$$

$$= y(1 + 0.5)$$

$$= y(1.5)$$

$$= y_0 + F(x_0, y_0) \cdot h$$

$$= 2 + F(1, 2) \cdot 0.5$$

$$= 2 + \{3 \times (1)^2 + 1\} \times 0.5 \quad [F(x, y) = 3x^2 + 1]$$

$$= 4$$

$$\therefore y(1.5) = 4 \Rightarrow x_1 = 1.5, y_1 = 4$$

Again,

$$y_2 = y(x_1 + h)$$

$$= y(1.5 + 0.5)$$

$$= y(2)$$

$$= y_1 + F(x_1, y_1) \cdot h$$

$$= 4 + \{3 \times (1.5)^2 + 1\} \times 0.5$$

$$= 7.875$$

$$y(2) = 7.875$$

① For $h = 0.25$

$$y_1 = y_0 + f(x_0, y_0) \cdot h$$

$$y(x_0 + h) = 2 + [f(1, 2) \times 0.25]$$

$$y(1 + 0.25) = 2 + [(3 \times 1) + 1] \times 0.25$$

$$y(1.25) = 3$$

$$\Rightarrow x_1 = 1.25, y_1 = 3$$

$$y_2 = y_1 + f(x_1, y_1) \cdot h$$

$$y(1.25 + 0.25) = 3 + f(1.25, 3) \times 0.25$$

$$y(1.5) = 3 + [3 \times (1.25)^2 + 1] \times 0.25$$

$$y(1.5) = 4.4218$$

$$\Rightarrow x_2 = 1.5, y_2 = 4.4218$$

$$y_3 = y_2 + f(x_2, y_2) \cdot h$$

$$y(1.5 + 0.25) = 4.4218 + [3 \times (1.5)^2 + 1] \times 0.25$$

$$y(1.75) = 6.3593$$

$$\Rightarrow y_3 = 6.359 \quad \text{and} \quad x_3 = 1.75$$

Finally,

$$y_4 = y_3 + F(x_3, y_3) \cdot h$$

$$y(1.75 + 0.25) = 6.3593 + [3 \times (1.75)^2 + 17] \times 0.25$$

$$\therefore y(2) = 8.9061$$

Heun's method (2nd order RK method)

This method uses two values of slopes m_1 & m_2 .

Putting $f(x_i, y_i) = m_1$ in Euler's method we get,

$$y_{i+1} = y_i + m_1 \cdot h \quad \text{--- (i) [for point } (x_i, y_i)]$$

$$y_{i+1} = y_i + m_2 \cdot h \quad \text{--- (ii) [for point } (x_{i+1}, y_{i+1})]$$

Heun's method states that,

$$y_{i+1} = y_i + \frac{m_1 + m_2}{2} \cdot h \quad \text{--- (iii)}$$

$$y_{i+1} = y_i + \frac{h}{2} [f(x_i, y_i) + f(x_{i+1}, y_{i+1})] \quad \text{From Euler's Formula}$$

$$y_{i+1} = y_i + \frac{h}{2} [f(x_i, y_i) + f(x_i + h, y_i + h f(x_i, y_i))]$$

Q. $y = \frac{2y}{x}$, $y(1) = 2$, Find $y(2)$ with $h = 0.25$.

Solⁿ Here,

$$y(1) = 2 \Rightarrow x_0 = 1, y_0 = 2$$

$$f(x, y) = y' = \frac{2y}{x}$$

Now,

Applying Heun's method,

$$y_{i+1} = y_i + \frac{h}{2} [F(x_i, y_i) + F(x_i + h, y_i + h F(x_i, y_i))]$$

$$y(1.25) = y_1 = y_0 + \frac{h}{2} [F(x_0, y_0) + F(x_1, y_0 + h F(x_0, y_0))]$$

$$= 2 + \frac{0.25}{2} [F(1, 2) + F(1.25, 2 + 0.25 \times F(1, 2))]$$

✓ Also

$$m_1 = F(x_i, y_i) = 2 + 0.25 \times \left[\frac{2 \times 2}{1} + F\left(1.25, 2 + 0.25 \times \frac{2 \times 2}{1}\right) \right]$$

$$m_2 = F(x_{th}, y_{th} * m_1)$$

$$y_{i+1} = y_i + \frac{h}{2} (m_1 + m_2) = 2 + 0.25 \times [4 + F(1.25, 3)]$$

$$= 2 + 0.25 \times \left[4 + \frac{2 \times 3}{1.25} \right]$$

$$= 3.1$$

$$\therefore y(1.25) = 3.1 \Rightarrow x_1 = 1.25, y_1 = 3.1$$

$$F(1.25, 3.1) + F(1.5, 1.25) + 0.25 F(1.25, 3.1)$$

Now,

$$y_2 = y(1.5) = y_1 + \frac{h}{2} [F(x_1, y_1) + F(x_2, y_1 + h F(x_1, y_1))]$$

$$= 3.1 + \frac{0.25}{2} \times \left[\frac{2 \times 3.1}{1.25} + F(1.5, 3.1 + 0.25 \times \frac{2 \times 3.1}{1.25}) \right]$$

$$= 3.1 + \frac{0.25}{2} \times [4.96 + F(1.5, 4.34)]$$

$$= 3.1 + \frac{0.25}{2} \left[4.96 + 2 \times 4.34 \right]_{1.5}$$

$$\therefore y(1.5) = 4.4433$$

$$\therefore x_2 = 1.5, y_2 = 4.4433$$

Again,

$$y_3 = y(1.5 + 0.25) = y(1.75)$$

$$= y_2 + \frac{h}{2} [F(x_2, y_2) + F(x_3, y_2 + hF(x_2, y_2))]$$

$$= 4.4433 + \frac{0.25}{2} [F(1.5, 4.4433)$$

$$+ F(1.75, 4.4433 + 0.25 \times F(1.5, 4.4433))]$$

$$= 4.4433 + \frac{0.25}{2} \left[2 \times 4.4433 + F(1.75, 4.4433 + \right.$$

$$\left. 0.25 \times 2 \times 4.4433 \right]_{1.5}$$

$$= 4.4433 + \frac{0.25}{2} [5.9244 + F(1.75, 5.924400)]$$

$$= 4.4433 + \frac{0.25}{2} [5.9244 + 2 \times 5.9244]_{1.75}$$

$$= 6.0302$$

$$y(1.75) = 6.0302$$

$$\Rightarrow x_3 = 1.75, y_3 = 6.0302$$

Finally,

$$y_4 = y(1.75 + 0.25) = y(2)$$

$$= y_3 + h \left[\frac{f(x_3, y_3) + f(x_4, y_3 + hf(x_3, y_3))}{2} \right]$$

$$= 6.0302 + \frac{0.25}{2} \left[f(1.75, 6.0302) + f(2, 6.0302 + 0.25 \times f(1.75, 6.0302)) \right]$$

$$= 6.0302 + \frac{0.25}{2} \left[\frac{2 \times 6.0302}{1.75} + f(2, 6.0302 + \frac{0.25 \times 2 \times 6.0302}{1.75}) \right]$$

$$= 6.0302 + \frac{0.25}{2} [6.8917 + f(2, 7.7531)]$$

$$= 6.0302 + \frac{0.25}{2} \left[6.8917 + \frac{2 \times 7.7531}{2} \right]$$

$$= 7.8608$$

$$\therefore y(2) = 7.8608$$

#

Picard's method

Consider the differential eqⁿ

$$y' = \frac{dy}{dx} = f(x, y) \quad \text{--- (1)}$$

with initial condition $y = y_0$ for $x = x_0$

Integrating the differential eqⁿ (1), we get

$$\int_{x_0}^{x_1} y' = \int_{x_0}^{x_1} f(x, y) \cdot dx$$

$$\text{or } y(x_1) - y(x_0) = \int_{x_0}^{x_1} f(x, y) \cdot dx$$

$$y(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x, y) \cdot dx \quad \text{--- (2)}$$

In eqⁿ (2), the 'y' appears under the integral sign, so integration can't be done.

Either we need to replace 'y' by constant or
function of x. We know that $y = y_0$ for $x = x_0$.

Thus, the first approximation y is obtained by replacing y by y_0 in $f(x, y)$ in R.H.S of eqⁿ (2) and

Integrating w.r.t. x , we get

$$y' = y_0 + \int_{x_0}^{x_1} f(x, y_0) \cdot dx$$

The second approximation y'' is obtained by replacing y by y' in $F(x, y)$ in R.H.S of eqⁿ (1) and Integrating w.r.t. x , we get

$$y'' = y_0 + \int_{x_0}^{x_1} F(x, y') \cdot dx$$

Proceeding in the same way, we obtain $y''', y^{(4)} \dots y^{(n-1)}$ only where,

$$y^{(n)} = y_0 + \int_{x_0}^{x_1} F(x, y^{(n-1)}) dx$$

with $y = y_0$

We repeat the steps till the two value of 'y' become same to the desired degree of accuracy.

Q

Solve the equation $y' = x^2 + y^2$ by using Picard's method with the initial condition $y(0) = 0$. Find the value of $y(0.2)$.

Solⁿ

Here,

$$y' = x^2 + y^2$$

$$y(0) = 0 \Rightarrow x_0 = 0, y_0 = 0$$

Since,

$$\begin{aligned} y' &= y_0 + \int_{x_0}^x F(x, y_0) \cdot dx \\ &= 0 + \int_0^x F(x, y_0) \cdot dx \end{aligned}$$

$$= \int_0^x f(x, y_0) \cdot dx$$

$$= \int_0^1 x^2 + (y^0)^2 \cdot dx = x^3/3$$

Since,

$$y'' = y_0 + \int_{x_0}^x f(x, y') \cdot dx$$

$$= 0 + \int_0^x f(xy') \cdot dx$$

$$= \int_0^x x^2 + (y')^2 \cdot dx$$

$$= \int_0^1 (x^2 + x^3/3) \cdot dx$$

$$= \frac{x^3}{3} + \frac{x^7}{63}$$

If we stop at y'' , then

$$y(x) = \frac{x^3}{3} + \frac{x^7}{63}$$

$$\therefore y(0.2) = \frac{(0.2)^3}{3} + \frac{(0.2)^7}{63}$$

$$= 0.00026667 \text{ } \text{\textcircled{D}}$$

4th method Runge-Kutta method (RK4)

These are four slopes in this method given as

Step 1 →	$m_1 = f(x_i, y_i)$
	$m_2 = f(x_i + h/2, y_i + m_1 \cdot h/2)$
	$m_3 = f(x_i + h/2, y_i + m_2 \cdot h/2)$
	$m_4 = f(x_i + h, y_i + m_3 \cdot h)$

and

Step 2 →	$y_{i+1} = y_i + \frac{h}{6} (m_1 + 2m_2 + 2m_3 + m_4)$
----------	---

Q

Solⁿ

$y'(x) = x^2 + y^2$, $y(0) = 0$, find $y(0.4)$ with $h = 0.2$
Here,

$$y(0) = 0 \Rightarrow x_0 = 0, y_0 = 0 \quad \text{and } h = 0.2$$

$$f(x, y) = x^2 + y^2$$

Now,

Iteration-1: For, $y_1 = y(0+0.2) = y(0.2)$

$$m_1 = f(x_0, y_0) = f(0, 0) = 0^2 + 0^2 = 0.$$

$$m_2 = f(x_0 + h/2, y_0 + m_1 \cdot h/2)$$

$$= f(0 + 0.2/2, 0 + 0 \cdot 0.2/2)$$

$$= f(0.1, 0) = (0.1)^2 + 0^2 = 0.01$$

Step 1

$$\begin{aligned}
 m_3 &= f(x_0 + h/2, y_0 + m_2 \cdot h/2) = f(0 + 0.2/2, 0 + 0.01 \times 0.2/2) \\
 &= f(0.1, 0.001) = (0.1)^2 + (0.001)^2 \\
 &= 0.010001
 \end{aligned}$$

$$\begin{aligned}
 m_4 &= f(x_0 + h, y_0 + m_3 \cdot h) = f(0 + 0.2, 0 + 0.010001 \times 0.2) \\
 &= f(0.2, 0.0020002) \\
 &= (0.2)^2 + (0.0020002)^2 = 0.040004
 \end{aligned}$$

Now,

$$\begin{aligned}
 \text{Step 2} \quad y_1 &= y_0(0 + 0.2) = y(0.2) \\
 &= y_0 + \frac{h}{6} (m_1 + 2m_2 + 2m_3 + m_4) \\
 &= 0 + 0.2 \frac{(0 + 2 \times 0.01 + 2 \times 0.010001 + 0.040004)}{6} \\
 &= 0.00267
 \end{aligned}$$

$$\therefore y(0.2) = 0.00267$$

$$\therefore x_1 = 0.2, y_1 = 0.00267$$

Iteration-2 :

$$\text{For, } y_2 = y(0.2 + 0.2) = y(0.4) = ?$$

$$m_1 = f(x_1, y_1) = f(0.2, 0.00267) = 0.04$$

$$\begin{aligned}
 m_2 &= f(x_1 + h/2, y_1 + m_1 \cdot h/2) \\
 &= f(0.2 + 0.2/2, 0.00267 + 0.04 \times 0.2/2) \\
 &= f(0.3, 0.0067) = 0.09005
 \end{aligned}$$

$$\begin{aligned}
 m_3 &= f(x_1 + h, y_1 + m_2 \cdot h/2) \\
 &= f(0.2 + 0.2/2, 0.00267 + 0.09005 \times 0.2/2) \\
 &= f(0.3, 0.0117) \\
 &= 0.0901
 \end{aligned}$$

$$\begin{aligned}
 m_4 &= f(x_1 + h, y_1 + m_3 \cdot h) = f(0.2 + 0.2, 0.00267 + 0.0901 \times 0.2) \\
 &= f(0.4, 0.0207) \\
 &= 0.1604
 \end{aligned}$$

Now,

$$y_2 = y(0.4)$$

$$= y_1 + \frac{h}{6} (m_1 + 2m_2 + 2m_3 + m_4)$$

$$= 0.00267 + \frac{0.2}{6} (0.04 + 0.09005 \times 2 + 2 \times 0.0901 + 0.1604)$$

$$= 0.02136$$

$$\therefore y(0.4) = 0.02136 \quad \#$$

#

Simultaneous first order differential eqⁿ

Simultaneous first order eqⁿ is of the form

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n) \quad \text{with } y_1(x_0) = y_{10}$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \dots, y_n) \quad \text{with } y_2(x_0) = y_{20}$$

⋮

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \dots, y_n) \quad \text{with } y_n(x_0) = y_{n0}$$

let us consider two simultaneous eqⁿ.

$$\frac{dy}{dx} = f(x, y, z) \quad \text{--- (I)} \quad \text{and} \quad \frac{dz}{dx} = g(x, y, z) \quad \text{--- (II)} \quad \text{with}$$

$$y(x_0) = y_0 \quad z(x_0) = z_0$$

then using RK4 method with slopes for eqⁿ (I) as ' m_i ' and slopes for eqⁿ (II) as ' l_i ', we have

$$m_1 = f(x_i, y_i, z_i)$$

$$l_1 = g(x_i, y_i, z_i)$$

$$m_2 = f(x_i + h/2, y_i + m_1 \cdot h/2, z_i + l_1 \cdot h/2)$$

$$l_2 = g(x_i + h/2, y_i + m_1 \cdot h/2, z_i + l_1 \cdot h/2)$$

$$m_3 = f(x_i + h/2, y_i + m_2 \cdot h/2, z_i + l_2 \cdot h/2)$$

$$l_3 = g(x_i + h/2, y_i + m_2 \cdot h/2, z_i + l_2 \cdot h/2)$$

$$m_4 = f(x_i + h, y_i + m_3 \cdot h, z_i + l_3 \cdot h)$$

$$l_4 = g(x_i + h, y_i + m_3 \cdot h, z_i + l_3 \cdot h)$$

$$y_{i+1} = y_0 + h/6 (m_1 + 2m_2 + 2m_3 + m_4)$$

$$z_{i+1} = z_0 + h/6 (l_1 + 2l_2 + 2l_3 + l_4)$$

As same in
RK4

- we can see that we need l_1 to find m_2 , m_2 to find l_2 , so, corresponding calculation will result in convenient way.

- And value in brackets are same so, $f(a, b, c) = g(a, b, c)$ so, the value depends upon function.

Ex

$$y' = x + z, \quad z' = x - y^2 \quad y(0) = 2, \quad z(0) = 1, \quad h = 0.1$$

Find $y(0.1)$ and $z(0.1)$.

Solⁿ

Here,

$$y(0) = 2$$

$$\Rightarrow x_0 = 0, y_0 = 2$$

$$z(0) = 1$$

$$\Rightarrow z_0 = 1, x_0 = 0$$

$$f(x, y, z) = y' = x + z$$

$$g(x, y, z) = z' = x - y^2$$

$$h = 0.1$$

Now,

$$m_1 = f(x_0, y_0, z_0)$$

$$= f(0, 2, 1)$$

$$= 0 + 1 = 1$$

$$l_1 = g(x_0, y_0, z_0)$$

$$= g(0, 2, 1)$$

$$= 0 - (2)^2 = -4$$

Euler

$$m_2 = f(x_0 + h/2, y_0 + m_1 \cdot h/2, z_0 + l_1 \cdot h/2)$$

$$= f(0 + 0.05, 2 + 1 \cdot 0.05, 1 + (-4) \cdot 0.05)$$

$$= f(0.05, 2.05, 0.8)$$

$$= 0.05 + 0.8 = 0.85$$

$$l_2 = g(0.05, 2.05, 0.8)$$

$$= 0.05 - (2.05)^2$$

$$= -4.1525$$

$$m_3 = f(x_0 + h/2, y_0 + m_2 \cdot h/2, z_0 + l_2 \cdot h/2)$$

$$= f(0 + 0.05, 2 + 0.85 \cdot 0.05, 1 - 4.1525 \cdot 0.05)$$

$$= f(0.05, 2.0425, 0.7924)$$

$$= 0.05 + 0.7924 = 0.8424$$

$$l_3 = g(0.05, 2.0425, 0.7924)$$

$$= 0.05 - (2.0425)^2$$

$$= -4.1218$$

$$\begin{aligned}
 m_4 &= f(x_0 + h, y_0 + m_3 \cdot h, z_0 + l_3 \cdot h) \\
 &= f(0 + 0.1, 2 + 0.8424 \times 0.1, 1 - 4.1218 \times 0.1) \\
 &= f(0.1, 2.08424, 0.58782) \\
 &= 0.1 + 0.58782 = 0.68782
 \end{aligned}$$

$$\begin{aligned}
 l_4 &= g(0.1, 2.08423, 0.58782) \\
 &= 0.1 - (2.08423)^2 \\
 &= -4.24401
 \end{aligned}$$

Now,

Finally,

$$\begin{aligned}
 y_1 &= y(0 + 0.1) = y(0.1) \\
 &= y_0 + \frac{h}{6} (m_1 + 2m_2 + 2m_3 + m_4) \\
 &= 2 + 0.1 \frac{(1 + 2 \times 0.85 + 2 \times 0.8424 + 0.68782)}{6} \\
 &= 2.08454
 \end{aligned}$$

And,

$$\begin{aligned}
 z_1 &= z(0 + 0.1) = z(0.1) \\
 &= z_0 + \frac{h}{6} (l_1 + 2l_2 + 2l_3 + l_4) \\
 &= 1 + 0.1 \frac{(-4 + 2 \times (-4.1525) + 2 \times (-4.1218) - 4.24401)}{6}
 \end{aligned}$$

$$= 0.58679$$

$$\therefore y(0.1) = 2.08454$$

$$\therefore z(0.1) = 0.58679$$

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#

Higher order EquationHigher order differential eqⁿ is of the form

$$\frac{d^m y}{dx^m}$$

with m initial condition given as:

$$y(x_0) = a_1$$

$$y'(x_0) = a_2$$

$$\vdots$$

$$y^{m-1}(x_0) = a_m$$

let us denote,

$$y = y_1$$

$$\frac{dy}{dx} = y_2$$

$$\frac{d^2 y}{dx^2} = y_3$$

$$\vdots$$

$$\frac{d^{m-1} y}{dx^{m-1}} = y_m$$

Then, we can write

$$\frac{dy_1}{dx} = y_2 \text{ with } y_1(x_0) = y_{10} = a_1$$

$$\frac{dy_2}{dx} = y_3 \text{ with } y_2(x_0) = y_{20} = a_2$$

$$\vdots$$

$$\frac{dy^{m-1}}{dx} = y_m \text{ with } y_{m-1}(x_0) = y_{(m-1)0} = a_{m-1}$$

$$\frac{dy_m}{dx} = \frac{d^m y}{dx^m} = f(x, y_1, y_2, \dots, y_m)$$

with $y_m(x_0) = y_{m0} = a_m$

This system is similar to the system of first order eqⁿs. Hence, we can solve this by any procedure applied for first order eqⁿ.

Q

Solve the following for $y(0.2)$ and $y'(0.2)$

$$\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 3y = 6x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6x - 2y' + 3y$$

S

Given,

$y(0) = 0, y'(0) = 1$. Use Runge's method.

solⁿ

let, $y = y_1$ and $\frac{dy}{dx} = y_2$

let, $h = 0.2$

z

same
for
all

$$f(x, y_1, y_2) = \frac{dy_1}{dx} = y_2 \quad \text{with} \quad y(0) = y_{10} = 0 \quad [\text{let slope} = m_1]$$

$$g(x, y_1, y_2) = \frac{dy_2}{dx} = 6x - 2y_2 + 3y_1 \quad \text{with} \quad y'(0) = y_{20} = 1 \quad [\text{let slope} = l_1]$$



Acc. to question

Now,

$$m_1 = f(x_0, y_{10}, y_{20}) = f(0, 0, 1) = 1 \quad \{6 \times 0 - 2 \times 1 - 2 \times 0\}$$

$$l_1 = g(x_0, y_{10}, y_{20}) = g(0, 0, 1) = -2$$

Similarly,

$$m_2 = f(x_0 + h, y_{10} + h m_1, y_{20} + h l_1) = f(0.2, 0.2, 0.6) = 0.6$$

$$l_2 = g(0.2, 0.2, 0.6) = 0.6$$

$$y_1(0.2) = y_1(0) + \frac{h}{2} (m_1 + m_2) = y_{10} + \frac{h}{2} (m_1 + m_2)$$

$$= 0 + \frac{0.2}{2} (1 + 0.6) = 0.16$$

$$y_2(0.2) = y_2(0) + \frac{h}{2} (l_1 + l_2) = y_{20} + \frac{h}{2} (l_1 + l_2)$$

$$= 1 + \frac{0.2}{2} (-2 + 0.6) = 0.86$$

$$\therefore \begin{cases} y(0.2) = 0.16 \\ y'(0.2) = 0.86 \end{cases}$$

Boundary value problems

→ In this method not only the initial conditions are given but values at both the boundaries (i.e. initial point & final point) are given.

• Shooting method

→ Boundary value problem is converted into initial value problem.

→ let, $y'' = F(x, y, y')$ with $y(a) = y(b) = A$
 $y(a) = y(b) = B$

Suppose $y' = z$

then $y' = z = f_1(x, y, z)$ and $z' = y'' = f_2(x, y, z)$

To convert Boundary value problem to initial value problem we need the conditions given as

①

$$y(a) = A$$

Now, we require another condition for z at $x=a$

let,

$$z(a) = z(a) = m$$

(just a guess)

represent the slope $y'(x)$ at $x=a$

$$y' = z = f_1(x, y, z) \text{ with } y(a) = A$$

$$y'' = z' = f(x, y, z) \text{ with } z(a) = M_1$$

⑪ Using, $z(a) = z(x_0) = m$

$$\text{Estimate } y(b) = y(x_n) = B$$

if $B = B$ (exact solution)

⑫ Else,

$$\text{Suppose } z(a) = z(x_0) = m_2 \text{ (another guess)}$$

$$\text{Estimate } y(b) = y(x_n) = B_2$$

if $B_2 = B$ (exact solⁿ)

Else estimate M using

$$\frac{M - M_1}{B - B_1} = \frac{M_2 - M_1}{B_2 - B_1}$$

Now, with $z(x_0) = z(a) = m_3$

We can obtain the solution of $y(x)$.

Q

$y'' = 6x^2$, $y(0) = 1$, $y(1) = 2$ in interval $(0, 1)$, $h = 0.5$, $y(0.5) = ?$.
 Take $z_0 = 0.5, 1.5$ as guesses. Use Euler's Method

Sol

Here,

$$z_0 = 0, y_0 = 1 = A \text{ (let)}$$

$$x_n = 1, y_n = 2 = B \text{ (let)}$$

let,

$$y' = z = f_1(x, y, z) \text{ then}$$

$$y'' = z' = f_2(x, y, z) = 6x^2$$

$$\text{let, } z_0 = 0.5 = M_1 \text{ (let)}$$

Now, Using Euler's method

$$y(0+0.5)$$

$$y_1 = y(0.5) = y_0 + h \cdot f_1(x_0, y_0, z_0) = 1 + 0.5 \times f_1(0, 1, 0.5)$$

$$= 1 + 0.5 \times 0.5 = 1.25$$

$$y(0.5) = 1.25 \Rightarrow z_1 = 0.5, y_1 = 1.25$$

$$z_1 = z(0.5) = z_0 + h \cdot f_2(x_0, y_0, z_0) = 0.5 + 0.18$$

$$y_2 = y(1) = y_1 + h \cdot f_1(z_1, y_1, z_1) = 1.25 + 0.5 \times f_1(0.5, 1.25, 0.5)$$

$$= 1.25 + 0.5 \times 0.5$$

$$= 1.5 = B_1 \text{ (let)}$$

Now,

$$\text{let, } z_0 = 1.5 = M_2 \text{ (let)}$$

Using Euler's method,

$$\begin{aligned} y_1 = y(0.5) &= y_0 + hf_1(x_0, y_0, z_0) = 1 + 0.5 \times f_1(0, 1, 1.5) \\ &= 1 + 0.5 \times 1.5 \\ &= 1.75 \end{aligned}$$

$$\therefore y(0.5) = 1.75 \Rightarrow x_1 = 0.5, y_1 = 1.75$$

$$z_1 = z(0.5) = z_0 + hf_2(x_0, y_0, z_0) = 1.5 + 0.5 \times 0 = 1.5$$

$$\begin{aligned} y_2 = y(1) &= y_1 + hf_1(x_1, y_1, z_1) = 1.75 + 0.5 \times f_1(0.5, 1.75, 1.5) \\ &= 1.75 + 0.5 \times 1.5 \\ &= 2.5 = B_2 \text{ (let)} \end{aligned}$$

Now,

let 'M' be the actual value of 'z', then

$$\frac{M - M_1}{B - B_1} = \frac{M_2 - M_1}{B_2 - B_1}$$

$$\text{or, } \frac{M - 0.5}{2 - 1.5} = \frac{1.5 - 0.5}{2.5 - 1.5}$$

$$\text{or, } M = 0.5 + 0.5 = 1 \quad \therefore M = 1$$

let, $z_0 = 1$

$$y_1 = y(0.5) = y_0 + h f_1(x_0, y_0, z_0) = 1 + 0.5 \times f_1(0, 1, 1) \\ = 1 + 0.5 \times 1 = 1.5$$

$$\therefore y(0.5) = 1.5 \Rightarrow x_1 = 0.5, y_1 = 1.5$$

$$z_1 = z(0.5) = z_0 + h f_2(x_0, y_0, z_0) = 1 + 0.5 \times 0 = 1$$

$$y_2 = y(1) = y_1 + h f_1(x_1, y_1, z_1) = 1.5 + 0.5 \times f_1(0.5, 1.5, 1) \\ = 1.5 + 0.5 \times 1 = 2$$

$$\therefore y(0.5) = 1.5 \text{ Ans}$$

⑥

#

Numerical Solution of Partial Differential Equations

The general form of 2nd order partial diff. eqⁿ is

$$a \frac{\partial^2 f}{\partial x^2} + b \frac{\partial^2 f}{\partial x \partial y} + c \frac{\partial^2 f}{\partial y^2} = f(x, y, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \quad \text{--- (1)}$$

If

a) $b^2 - 4ac < 0$ eqⁿ (1) is called elliptic eqⁿ

b) $b^2 - 4ac = 0$ " " " " parabolic "

c) $b^2 - 4ac > 0$ " " " " hyperbolic "

① Solution of elliptic eqⁿ

When $a=1, b=0$ and $c=1$, eqⁿ (1) becomes

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f(x, y, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \quad \text{--- (1)}$$

① if $\nabla^2 f = 0$ eqⁿ (1) is called Laplace eqⁿ

② if $\nabla^2 f \neq 0$ " " " " Poisson's eqⁿ

Solution of Laplace eqⁿ by finite difference method.

From Taylor's series method, we have the difference formula as.

$y' = \frac{y_{i+1} - y_i}{h}$
$y'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$

It is also known as Finite difference formula.

Now, we have the Laplace eqⁿ given as

$$\nabla^2 F = 0$$

$$\text{or, } \frac{\partial^2 F(x, y)}{\partial x^2} + \frac{\partial^2 F(x, y)}{\partial y^2} = 0$$

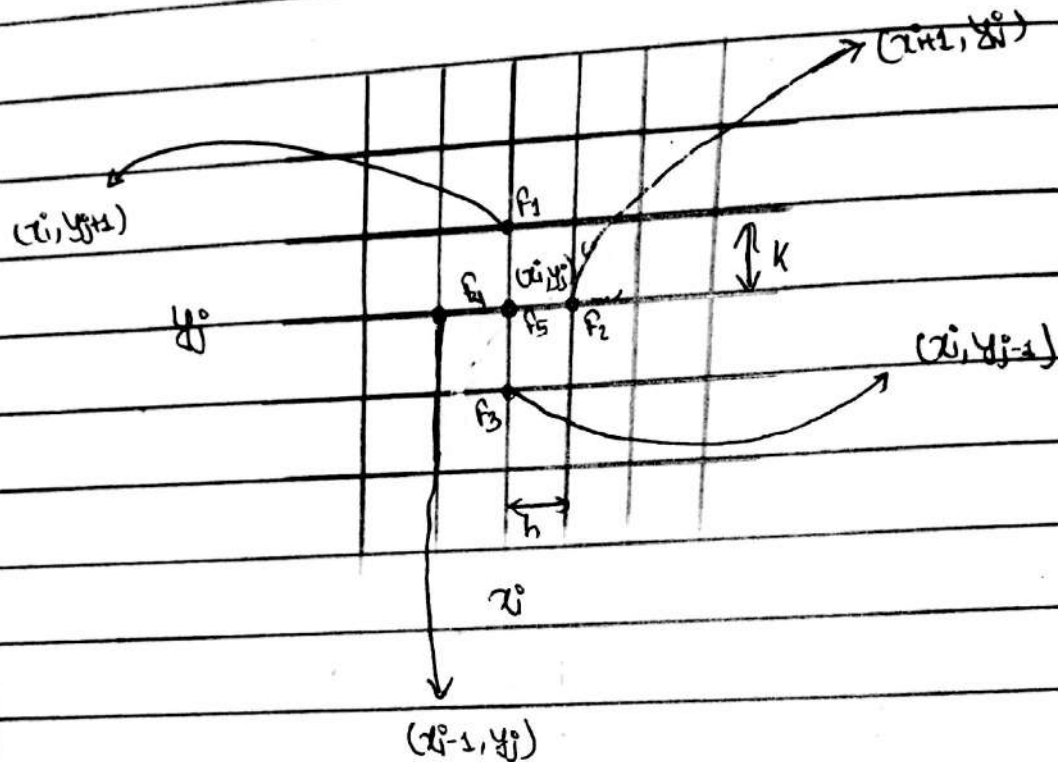
Applying the finite difference formula to Laplace eqⁿ, we get

$$\frac{\partial^2 F(x, y)}{\partial x^2} + \frac{\partial^2 F(x, y)}{\partial y^2} = 0$$

$$\text{or, } \frac{F(x_{i+1}, y) - 2F(x_i, y) + F(x_{i-1}, y))}{h^2} + \frac{F(x, y_{j+1}) - 2F(x, y_j) + F(x, y_{j-1}))}{h^2}$$

$$= 0$$

$(x_{i+1}, y_j) \rightarrow (x_i + h, y_j)$
Same



$$\text{or, } \frac{f_2 - 2f_5 + f_4}{h^2} + \frac{f_1 - 2f_5 + f_3}{k^2} = 0$$

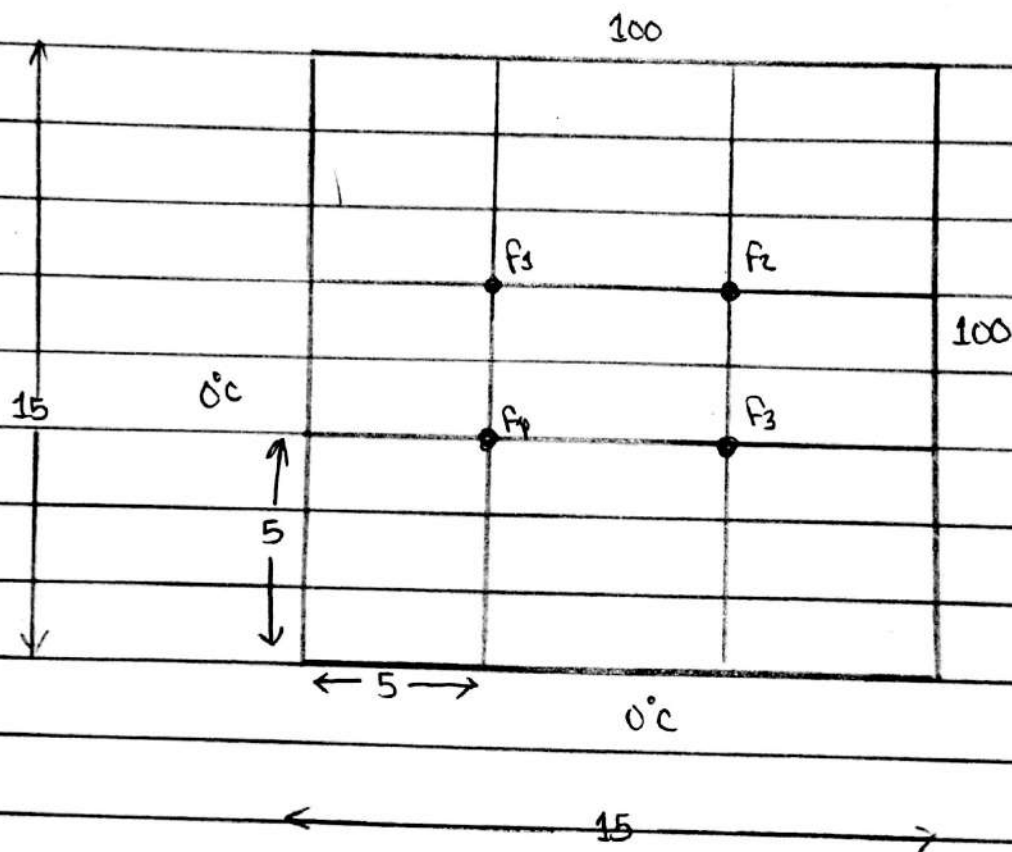
for $h=k$, we have,

$$\frac{1}{h^2} [f_2 - 2f_5 + f_4 + f_1 - 2f_5 + f_3] = 0$$

$$\text{or, } f_1 + f_2 + f_3 + f_4 - 4f_5 = 0$$

which is known as 5-point formula

Q. Consider a steel plate of size 15×15 cm. If two adjacent sides are held at 0°C and other two sides are at 100°C . What are the steady state value of temperature at interior nodes. Dimension of each grid is 5×5 cm. Assume Laplace distribution.



Applying five point Laplace formula for f_1 as reference point

$$0 + 100 + f_4 + f_2 - 4f_1 = 0 \quad \text{--- (I)}$$

Similarly, $100 + f_3 + f_1 + 100 - 4f_2 = 0 \quad \text{--- (II)}$

$$100 + 0 + f_4 + f_2 - 4f_3 = 0 \quad \text{--- (III)}$$

$$0 + 0 + f_3 + f_1 - 4f_4 = 0 \quad \text{--- (IV)}$$

• Easiest way to solve :-

- ① Take one eqⁿ and take any one of the variable in one side, next to other
- ② Use the value in next two eq^s
- ③ Don't touch one eqⁿ

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Date _____
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2 eqⁿ
3 eqⁿ solve this

~~From ①~~

Solving eqⁿ ①, ②, ③ & ④, we get

From ①,

$$f_1 = 4f_2 - f_3 - 200 \quad \text{--- ②}$$

From ①,

$$100 + f_1 + f_2 - 4(4f_2 - f_3 - 200) = 0$$

$$\text{or, } 100 + f_1 + f_2 - 16f_2 + 4f_3 + 800 = 0$$

$$\text{or, } -15f_2 + 4f_3 + f_1 = -900 \quad \text{--- ⑥}$$

From ④,

$$f_3 + (4f_2 - f_3 - 200) - 4f_4 = 0$$

$$\text{or, } f_3 + 4f_2 - f_3 - 200 - 4f_4 = 0 \quad \text{--- ⑦}$$

$$\text{or, } 4f_2 - 4f_4 = 200$$

Eq^s are :-

$$-15f_2 + 4f_3 + f_1 = -900 \quad (\text{from ⑥})$$

$$4f_2 + 0 \cdot f_3 - 4f_4 = 200 \quad (\text{from ⑦})$$

$$f_2 - f_4 = 50 \quad (\text{from ⑦})$$

By solving

$$\therefore f_2 = 75$$

$$f_3 = 50$$

$$f_4 = 25$$

Substituting the value of f_2, f_3, f_4 in eqn (a) to find the value of f_1 .

$$\begin{aligned}\therefore f_1 &= 4f_2 - f_3 - 200 \\ &= 4 \times 75 - 50 - 200 \\ &= -50\end{aligned}$$

Answers are:-

Intensities temperature

$$\Rightarrow f_1 = 50$$

$$f_2 = 75$$

$$f_3 = 50$$

$$f_4 = 25$$

} #

Solution of Poisson's eqⁿ by Finite difference Method

the Poisson's eqⁿ is of the form

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = g(x, y) \quad \text{where, } g(x, y) \neq 0$$

From Taylor's series method,

$$y' = \frac{y_{i+1} - y_i}{h}$$

$$y'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Now, we have Poisson eqⁿ given as

$$\nabla^2 f = g(x, y) \Rightarrow \frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} = g(x, y)$$

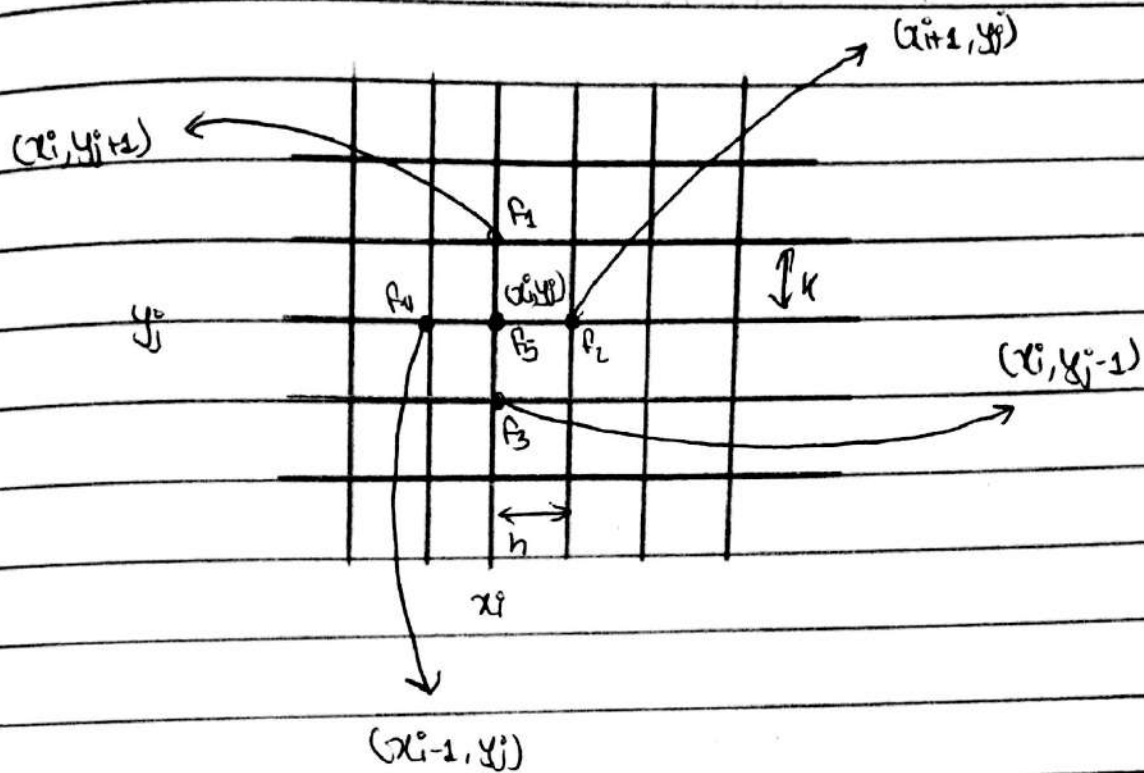
Applying finite difference formula to ^{Poisson} Laplace eqⁿ, we get

$$\frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} = g(x, y)$$

$$\text{or, } \frac{f(x_{i+1}, y_i) - 2f(x_i, y_i) + f(x_{i-1}, y_i))}{h^2}$$

$$+ \frac{f(x_i, y_{j+1}) - 2f(x_i, y_j) + f(x_i, y_{j-1}))}{k^2} = g(x, y)$$

$\begin{matrix} i \\ \nearrow \\ j \end{matrix}$



$$\text{or, } \frac{f_2 - 2f_5 + f_4}{h^2} + \frac{f_1 - 2f_5 + f_3}{k^2} = g(x, y)$$

For $h=k$, we have

$$\frac{1}{h^2} [f_2 - 2f_5 + f_4 + f_1 - 2f_5 + f_3] = g(x, y)$$

$$\therefore f_1 + f_2 + f_3 + f_4 - 4f_5 = h^2 \cdot g(x, y)$$

where $h=k$.

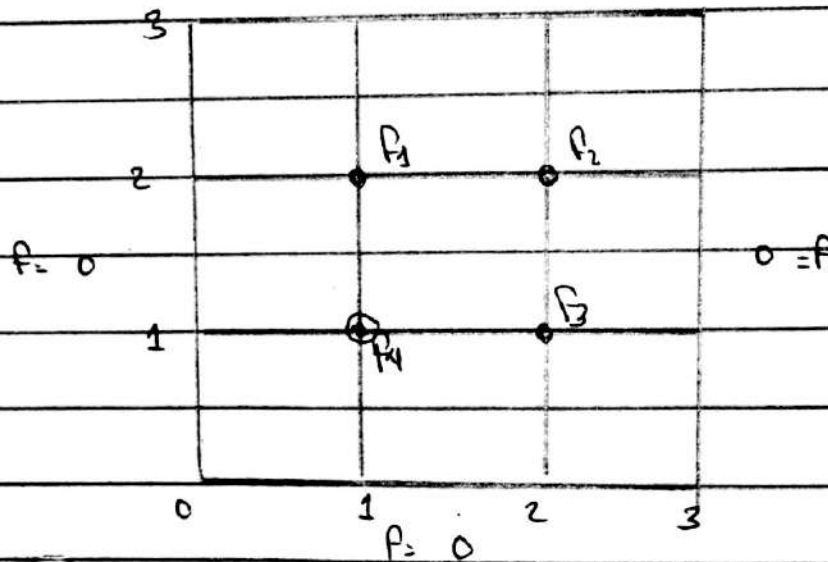
Q

$\nabla^2 f = 2xy^2$ over the square domain $0 \leq x \leq 3$ and $0 \leq y \leq 3$ with $f=0$ on the boundary and $h=k=1$. Find the values at interior nodes.

Solⁿ

Here,

$$g(x,y) = 2xy^2$$

 $f = 0$ 

Applying five point formula for P_1 as reference point.

$$f_1 + f_3 + 0 + 0 - 4f_4 = 2 \times 1^2 \cdot 1^2$$

$$\Rightarrow f_1 + f_3 - 4f_4 = 2 \quad \text{--- (i)}$$

Similarly,

$$f_2 + f_4 + 0 + 0 - 4f_3 = 2 \times 1^2 \cdot 2^2$$

$$\Rightarrow f_2 + f_4 - 4f_3 = 8 \quad \text{--- (ii)}$$

$$f_1 + f_3 + 0 + 0 - 4f_2 = 2 \times 2^2 \cdot 2^2$$

$$\Rightarrow f_1 + f_3 - 4f_2 = 32 \quad \text{--- (iii)}$$

$$f_2 + f_4 + 0 + 0 - 4f_1 = 2 \cdot 2^2 \cdot 1^2$$

$$\Rightarrow f_2 + f_4 - 4f_1 = 8 \quad \text{--- (v)}$$

Solving eqⁿ (i), (ii) & (v), we get.

From (i)

$$f_1 = 2 + 4f_4 - f_3 \quad \text{--- (vi)}$$

From (ii)

$$f_2 + f_4 - 4(2 + 4f_4 - f_3) = 8$$

$$\text{or, } f_2 + f_4 - 8 - 16f_4 + 4f_3 = 8$$

$$\text{or, } f_2 - 15f_4 + 4f_3 - 8 = 8$$

$$\therefore f_2 - 15f_4 + 4f_3 = 16 \quad \text{--- (vii)}$$

Again,

From (iii)

$$(2 + 4f_4 - f_3) + f_3 - 4f_2 = 32$$

$$\text{or, } 2 + 4f_4 - f_3 + f_3 - 4f_2 = 32$$

$$\therefore 4f_4 - 4f_2 = 30 \quad \text{--- (viii)}$$

Eqⁿs are

$$f_2 - 4f_3 + f_4 = 8 \quad \text{(from (v))}$$

$$f_2 + 4f_3 - 15f_4 = 16$$

$$-4f_2 + 4f_4 = 30$$

$$\therefore f_2 = -10.75, f_3 = -5.5, f_4 = 3.25$$

From (vi)

$$f_1 = 2 + 4 \times 3.25 + 5.5 = -5.5$$

$$\therefore f_1 = -5.5, f_2 = -10.75, f_3 = -5.5 \text{ and } f_4 = 3.25$$